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EXERCISES

1.1 For two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d$, the line through $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ can be written as: $\{\boldsymbol{x} | \boldsymbol{x} = \boldsymbol{x}_1 + \lambda(\boldsymbol{x}_2 - \boldsymbol{x}_1), \quad \lambda \in \mathbb{R}\}$. Equivalently, we can define the line as:

$$\{\boldsymbol{x} | \boldsymbol{x} = (1 - \lambda)\boldsymbol{x}_1 + \lambda \boldsymbol{x}_2, \quad \lambda \in \mathbb{R}\},$$

or

$$\{\boldsymbol{x} | \boldsymbol{x} = p_1 \boldsymbol{x}_1 + p_2 \boldsymbol{x}_2, \quad p_1, p_2 \in \mathbb{R}, \quad p_1 + p_2 = 1\}.$$