# Curvature Analysis in Complex Networks: Theory and Application

by

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I dedicate this thesis to my beloved parents, Fatemeh Tafreshian and MohammadReza Yahyanejad for their infinite supports and encouragements to pursue my dreams. To my dearest siblings: Faezeh and Meysam whom love has always kept my heart warm from thousands of miles away. To my best friend, Dr. Mohammad Ahmadpoor, who has always been there for me through my ups and downs, and to my beloved Dr. Emad Ghadirian, for his unconditional love and support.

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# PREFACE

This thesis is based on the following publications:

- Bhaskar DasGupta, Marek Karpinski, Nasim Mobasheri, and Farzane Yahyanejad, *Effect of Gromov-hyperbolicity Parameter on Cuts and Expansions in Graphs and Some Algorithmic Implications*, Algorithmica, 80(2), 772-800, 2018.
- Bhaskar DasGupta, Mano Vikash Janardhanan, and Farzane Yahyanejad, *How Did the Shape of Your Network Change? (On Detecting Anomalies in Static and Dynamic Networks via Change of Non-local curvatures)*, arXiv:1808.05676 [cs.DS], 2018.

## TABLE OF CONTENTS

## TABLE OF CONTENTS (Continued)

## LIST OF FIGURES

## LIST OF FIGURES (Continued)

**FIGURE** **PAGE**

# LIST OF ABBREVIATIONS

| | |
|---|---|
| DkS | Densest k Subgraph |
| EADP | Extremal Anomaly Detection Problem |
| EHSSC | Edge Hitting Set Sized Constrained |
| MSE | Minimum Shared Edge |
| SSE | Small Set Expansion |
| UGC | Unique Games Conjecture |
| UUMV | Unweighted Uncapacitated Minimum Vulnerability |

# SUMMARY

Useful insights for many complex systems are often obtained by representing them as networks and analyzing them using algorithmic tools and network measures (i.e., "network curvature"). In this thesis, we will use (a) Gromov-hyperbolic combinatorial curvature $\delta$ based on the properties of exact and aproximate geodesics distributions and higher-order connectivities and (b) Geometric curvatures $\mathfrak{C}$ based on identifying network motifs with geometric complexes ("geometric motifs" in systems biology jargon). Gromov-hyperbolic curvature in graphs occur often in many network applications. When the curvature is fixed, such graphs are simply called hyperbolic graphs and include non-trivial interesting classes of "non-expander" graphs. In this thesis we investigate the effect of $\delta$ on expansion and cut-size bounds on graphs (here $\delta$ need not be a constant), and the asymptotic ranges of $\delta$ for which these results provide improved approximation algorithms for related combinatorial problems, minimizing bottleneck edges problem and Small-set expansion problem. To this effect, we provide constructive bounds on node expansions for $\delta$-hyperbolic graphs as a function of $\delta$, and show that many witnesses (subsets of nodes) for such expansions can be computed efficiently. We also formulate and analyze geometric curvature based on defining k-complex-based Formans combinatorial Ricci curvature for elementary components, and using Euler characteristic of the complex that is topologically associated with the given graph. Since, $\mathfrak{C}$ depends on non-trivial global properties, we formulate several computational problems related to anomaly detection in static networks, and provide non-trivial computational complexity results for these problems.

# CHAPTER 1

# INTRODUCTION

The analysis, discrimination, and synthesis of complex networks rely on the use of measurements capable of expressing the most relevant topological features. Complex systems such as the world-wide web, social networks, metabolic networks, and protein-protein interaction networks can often be obtained by representing them as parameterized networks and analyzing them using graph-theoretic tools. In principle, we can classify these networks into two major classes:

- Static networks that model the corresponding system by one fixed network. Examples of such networks include biological signal transduction networks without node dynamics, and most social networks.
- Dynamic networks where elementary components of the network (such as nodes or edges) are added and/or removed as the network evolves over time. Examples of such networks include biological signal transduction networks with node dynamics, causal networks reconstructed from DNA microarray time-series data, biochemical reaction networks and dynamic social networks. Typically, such networks may have so-called *critical* (*elementary*) *components* whose presence or absence alters some significant *non-trivial non-local* property of these networks. For example:

- ▷ For a *static network*, there is a rich history in finding various types of critical components dating back to quantifications of *fault-tolerance* or *redundancy* in electronic circuits or routing networks. Recent examples of practical application of determining critical and non-critical components in the context of systems biology include quantifying *redundancies* in biological networks (52; 68; 5) and confirming the existence of *central influential* neighborhoods in biological networks (2).
- ▷ For a *dynamic network*, critical components may correspond to a set of nodes or edges whose addition and/or removal *between two time steps* alters a significant topological property of the network. Popularly also known as the *anomaly detection* or *change-point detection* (7; 49) problem, these types of problems have been studied over the last several decades in data mining, statistics and computer science *mostly in the context of time series data* with applications to areas such as medical condition monitoring (76; 14), weather change detection (29; 61) and speech recognition (20; 64).

## 1.1 Why use of network curvature?

Prior researchers have proposed and evaluated a number of established network measures such as *degree-based measures* (*e*.g., degree distribution), *connectivity-based measures* (*e*.g., clustering coefficient), *geodesic-based measures* (*e*.g., betweenness centrality) and other more novel network measures (21; 53; 5; 9) for analyzing networks. The network measures considered in this thesis are "appropriate notions" of *network curvatures*. As provably demonstrated in published research works such as (2; 74; 73; 66), these network curvature measures saliently encode *non-trivial higher-order correlation* among nodes and edges that *cannot* be obtained by other popular network measures. Some important characteristics of these curvature measures that we consider are (2, Section (III))(48):

- These curvature measures depend on *non-trivial global* network properties, as opposed to measures such as *degree distributions* or *clustering coefficients* that are *local* in nature or *dense subgraphs* that use *only* pairwise correlations.
- These curvature measures can mostly be computed efficiently in polynomial time, as opposed to measures such as *community decompositions*, *cliques* or *densest-k-subgraphs.*
- When applied to real-world biological and social networks, these curvature measures can explain *many phenomena one frequently encounters in real network applications that are not easily explained by other measures* such as:
  - paths mediating up- or down-regulation of a target node starting from the same regulator node in *biological regulatory networks* often have many small crosstalk paths, and
  - existence of congestions in a node that is not a hub in *traffic networks.*

  Further details about the suitability of our curvature measures for real biological or social networks are provided in Section 2.3 for Gromov-hyperbolic curvature and Section 6.1.0.2 for geometric curvatures.

Curvatures are very natural measures of anomaly of higher dimensional objects in mainstream physics and mathematics (15; 11). However, networks are *discrete objects* that *do not neces-*

*sarily have an associated natural geometric embedding.* In our research we seek to adapt the definition of curvature from the non-network domains in a suitable way for detecting network anomalies. For example, in networks with sufficiently small Gromov-hyperbolicity and sufficiently large diameter a suitably small subset of nodes or edges can be removed to stretch the geodesics between two distinct parts of the network by an exponential amount leading to extreme implications on the expansion properties of such networks (10; 25), which is akin to the characterization of singularities (an extreme anomaly) by geodesic incompleteness (*i.e.*, stretching all geodesics passing through the region infinitely) (40).

## 1.2 Two notions of graph curvature (Scalar vs. vector curvature)

In this thesis, a *curvature* for a graph $G$ is a *scalar-valued* function $\mathfrak{C} \stackrel{\text{def}}{=} \mathfrak{C}(G) : G \mapsto \mathbb{R}$. The standard Gromov-hyperbolic curvature measure $\mathfrak{C}_{Gromov}(G) = \delta$ is always a scalar value. Geometric curvatures however could also be defined by a vector by looking at local curvatures at all *elementary components* (*e*.g., nodes or edges) of a network, and defining the overall curvature as a *vector* of these values. There are several ways in which network curvature can be defined depending on the type of global properties the measure is desired to affect; in this thesis we consider two such definitions as described subsequently. We leave algorithmic analysis of such geometric vector curvatures, which seems to require considerably different combinatorial and optimization tools, for future research. Both scalar and vector versions of curvatures are used in physics and mathematics to study higher-dimensional objects with their own pros and cons. For example, for a two-dimensional curve, the standard curvature as defined by Cauchy is a scalar curvature whereas the normal vector used in the study of differential geometry of curves is a vector curvature. Even though a casual glance may seem to suggest that the scalar curvature is a weak concept with inadequate influence on the global geometry of the higher-dimensional object that is being studied, there exists non-trivial results (*e*.g., the positive mass theorem of Schoen, Yau and Witten) that suggest that this may *not* be the case.

## 1.3 Organization of this Thesis

In this thesis, first we look into Gromov-hyperbolic curvature denoted by $\delta$ which reflects how the metric space (distances) of a graph is close to the metric space of a tree. Therefore, Chapter 2 is dedicated to provide Gromov-hyperbolic curvature definition and relevant known results about this topological property. In Chapter 3, We are motivated to investigate the effect of the hyperbolicity measure $\delta$ on expansion and cut-size bounds on graphs where $\delta$ is a free parameter and not necessarily a constant. These bounds can be used to obtain improved approximation algorithms for related combinatorial problems depends on the values of $\delta$. Since arbitrarily large $\delta$ leads to the class of all possible graphs, our hope is that investigations of this type will provide a characterization of hard graph instances for combinatorial problems via a lower bound on $\delta$. To this effect, in this chapter we further investigate the non-expander properties of hyperbolic networks beyond what is shown in (83; 102) and provide constructive proofs of witnesses (subsets of nodes) of small expansion or small cut-size.

The major motivation of this chapter is to address research questions of the following generic nature:

*"What is the effect of the hyperbolicity measure $\delta$ on expansion and cut-size bounds on graphs (where $\delta$ is a free parameter and not a necessarily a constant)? For what asymptotic*

*ranges of values of $\delta$ can these bounds be used to obtain improved approximation algorithms for related combinatorial problems?"*

Chapter 4 and chapter 5 provide algorithmic consequences of constructive bounds that found in previous chapter and related proof techniques for two problems, minimizing bottleneck edge problem and small set expansion problem respectively.

Then, In Chapter 6 we focus on geometric curvature denoted by $\mathfrak{C}$ and some remarks about this suitable measure for real-network. This chapter provides the foundations of systematic approaches to find *critical components* and *detect anomalies* in networks.

In Chapter 7, we desire to research on questions of the following *generic* type:

> *"Given a static network, identify the critical components of the network that "encode" significant non-trivial global properties of the network".*

To identify critical components, one first needs to provide details for following four specific items:

***(i)*** *network model selection*,

***(ii)*** *definition of elementary critical components*, and

***(iii)*** *network property selection* (*i.e.*, the global properties of the network to be investigated).

The specific details for these items in this research are as follows:

***(i)*** **Network model selection:** Our network model will be undirected graphs.

***(ii)*** **Critical component definition:** Individual edges are elementary members of critical components.

***(iii)*** **Network property selection:** The network measure for this research will be appropriate notions of "network curvature". More specifically, in this part we will analyze **geometric curvatures** *based on identifying network motifs with geometric complexes* ("geometric motifs" in systems biology jargon) *and then using Forman's combinatorializations.*

Finally in Chapter 8, we conclude with some interesting future research paths of our work.

# CHAPTER 2

# GROMOV-HYPERBOLIC CURVATURE

In this chapter we consider a topological measure called Gromov-hyperbolicity (or, simply hyperbolicity for short) for undirected unweighted graphs that has recently received significant attention from researchers in both the graph theory and the network science community. This hyperbolicity measure $\delta$ was originally conceived in a somewhat different group-theoretic context by Gromov (97). The measure was first defined for infinite continuous metric space via properties of geodesics (87), but was later also adopted for finite graphs. Lately, there have been a surge of theoretical and empirical works measuring and analyzing the hyperbolicity of networks, and many real-world networks, such as the following, have been reported (either theoretically or empirically) to be $\delta$-hyperbolic for $\delta = O(1)$:

- "preferential attachment" scale-free networks with appropriate scaling(normalization) (98),
- networks of high power transceivers in a wireless sensor network (79),
- communication networks at the IP layer and at other levels (107), and
- an assorted set of biological and social networks (2).

## 2.1 Formal Definitions of Gromov-hyperbolicity

There are several equivalent definitions (up to a multiplicative constant) of Gromov's hyperbolic metric space (79). The following definitions are the hyperbolicity measure via geodesic triangles and 4-node conditions which we use in this thesis.

**Definition 1** ($\delta$-hyperbolic graphs via geodesic triangles)**.** *A graph G has a (Gromov) hyperbolicity of $\delta = \delta(G)$, or simply is $\delta$-hyperbolic, if and only if for every three ordered triple of shortest paths $(\overline{u,v}, \overline{u,w}, \overline{v,w})$, u,v lies in a $\delta$-neighborhood of $\overline{u,w} \cup \overline{v,w}$, i.e., for every node x on $\overline{u,v}$, there exists a node y on $\overline{u,w}$ or $\overline{v,w}$ such that $dist_G(x,y) \leq \delta$. A $\delta$-hyperbolic graph is simply called a hyperbolic graph if $\delta$ is a constant.*

**Definition 2** (the class of hyperbolic graphs)**.** *Let G be an infinite collection of graphs. Then, $\mathfrak{G}$ belongs to the class of hyperbolic graphs if and only if there is an absolute constant $\delta \geq 0$ such that any graph $G \in \mathfrak{G}$ is $\delta$-hyperbolic. If $\mathfrak{G}$ is a class of hyperbolic graphs then any graph $G \in \mathfrak{G}$ is simply referred to as a hyperbolic graph. There is another alternate but equivalent (?up to a constant multiplicative factor?) way of defining $\delta$-hyperbolic graphs via the following 4-node conditions.*

**Definition 3** (equivalent definition of $\delta$-hyperbolic graphs via 4-node conditions)**.** *For a set of four nodes $u_1, u_2, u_3, u_4$, let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be a permutation of $\{1, 2, 3, 4\}$ denoting a rearrangement of the indices of nodes such that*

$$S_{u_1,u_2,u_3,u_4} = dist_{u_{\pi_1},u_{\pi_2}} + dist_{u_{\pi_3},u_{\pi_4}} \leq M_{u_1,u_2,u_3,u_4} = dist_{u_{\pi_1},u_{\pi_3}} + dist_{u_{\pi_2},u_{\pi_4}}$$

$$\leq L_{u_1,u_2,u_3,u_4} = dist_{u_{\pi_1},u_{\pi_4}} + dist_{u_{\pi_2},u_{\pi_3}}$$

*and let* $\rho_{u1,u2,u3,u4} = \frac{L_{u1,u2,u3,u4} - M_{u1,u2,u3,u4}}{2}$.

*Then, G is $\delta$-hyperbolic if and only if $\delta = max_{u_1,u_2,u_3,u_4 \in V}\{\rho_{u_1,u_2,u_3,u_4}\}$.*

There is a constant $c > 0$ such that if a graph G is $\delta$-hyperbolic and $\delta_2$-hyperbolic via Definition 1 and Definition 3, respectively, then $\frac{1}{c}\delta_1 \leq \delta_2 \leq c\delta_1$. Then, Definition 1 and Definition 3 are equivalent in the sense that they are related by a constant multiplicative factor (87). Since constant factors are not optimized in our proofs, we will use either of the two definitions of hyperbolicity in the sequel as deemed more convenient. Using Definition 3 and casting the resulting computation as a (max,min) matrix multiplication problem allows one to compute $\delta$ and a 2-approximation of $\delta$ in $O(n^{3.69})$ and in $O(n^{2.69})$ time, respectively (94). Several routing-related problems or the diameter estimation problem become easier if the network is hyperbolic (88; 89; 90; 96). For a discussion of properly scaled 4-node conditions that yield a variety of (non necessarily hyperbolic) geometries, see (99).

## 2.2 Remarks on Topological Characteristics of Hyperbolicity Measure

In this section, we present some non-trivial topological characteristics of Hyperbolicity Measure, even though the hyperbolicity property is often referred to as a "tree-like" property. For example three following examples of these characteristics are:

- **The hyperbolicity property is not hereditary (and thus also not monotone).** In Fig.1 we can see that removing a single node or edge can increase/decrease the value of $\delta$ abruptly.

- **"Close to hyperbolic topology" is not necessarily the same as "close to tree topology".** For example, all bounded-diameter graphs are also hyperbolic graphs irrespective of whether they are tree or not (however, hyperbolic graphs need not be of bounded diameter). In general, even for small $\delta$, the metric induced by a $\delta$-hyperbolic graph may be quite far from a tree metric (88).

- **Hyperbolicity is not necessarily the same as tree-width.** A somewhat related similar popular measure used in both the bioinformatics and theoretical computer science literature is the treewidth measure first introduced by Robertson and Seymour (109). Many NP-hard problems on general networks in fact allow polynomial-time solutions if restricted to classes of networks with bounded treewidth (86). However, as observed in (103) and elsewhere, the two measures are quite different in nature.

When $\delta$ is a constant, we can categorize trees, chordal graphs, cactus of cliques, AT-free graphs, link graphs of simple polygons, and any class of graphs with a fixed diameter as hyperbolic graph classes and when $\delta$ is not a constant, examples of non-hyperbolic graph classes include expanders, simple cycles, and, for some parameter ranges, the Erdos-Renyi random graphs. In this research, to avoid division by zero in terms involving $1/\delta$, we will assume $\delta > 0$. In other words, we will treat a 0-hyperbolic graph (a tree) as a 1-hyperbolic graph in the analysis.

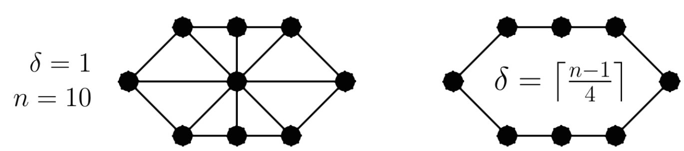


Figure 1: Hyperbolicity is not a hereditary property.

## 2.3 Relevant Known Results for Gromov Hyperbolicity

Here we summarize relevant known results that are used in this paper below; many of these results appear in several prior works, e.g., (2; 83; 87; 97; 103).

**Fact 1 (Cylinder removal around a geodesic).** *(102) Assume that $G$ is a $\delta$-hyperbolic graph. Let $p$ and $q$ be two nodes of $G$ such that $dist_G(p,q) = \beta > 6$, and let $p'$,$q'$ be nodes on a shortest path between $p$ and $q$ such that $dist_G(p,p') = dist_G(p',q') = dist_G(q',q) = \beta/3$. For* *any* $0 < \alpha < 1/4$, *let* $C$ *be set of nodes at a distance of* $\alpha\beta - 1$ *of a shortest path* $\overline{p', q'}$ *between* $p'$ *and* $q'$, *i.e., let* $C = \{u | \exists v \in \overline{p', q'} : dist_G(u, v) = \alpha\beta - 1\}$. *Let* $G_{-C}$ *be the graph obtained from* $G$ *by removing the nodes in* $C$. *Then,* $dist_{G_{-C}}(p, q) \geq (\beta/60)2^{\alpha\beta/\delta}$.

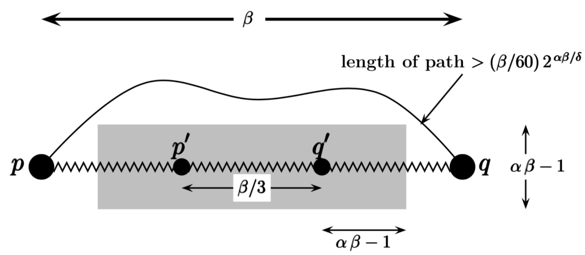


Figure 2: Illustration of Fact 1. By growing the shaded region and removing nodes in its boundary, one can selectively extract longer paths in the graph (i.e., the length of a shortest path between p and q increases when the nodes in the boundary of the shaded region are removed and the increase of the length of such a shortest path is more the larger the shaded region is). Translating the region slightly does not change this property much.

Fig. 2 pictorially illustrates this result.

**Fact 2** (**Exponential divergence of geodesic rays**). *[Simplified reformulation of (2), Theorem 10] Assume that* $G$ *is a* $\delta$*-hyperbolic graph. Suppose that we are given the following:*

- *three integers* $\kappa \geq 4, \alpha > 0, r > 3\kappa\delta$, *and*
- *five nodes* $v, u_1, u_2, u_3, u_4$ *such that* $dist_G(v, u_1) = dist_G(v, u_2) = r, dist_G(u_1, u_2) \geq 3\kappa\delta$, $dist_G(v, u_3) = dist_G(v, u_4) = r + \alpha$, *and* $dist_G(u_1, u_4) = dist_G(u_2, u_3) = \alpha$.

*Consider any path $Q$ between $u_3$ and $u_4$ that does not involve a node in $\bigcup_{0 \le j \le r+\alpha} B_G(v, j)$. Then, the length $|Q|$ of the path $Q$ satisfies $|Q| > 2^{\alpha/(6\delta)+\kappa+1}$.*

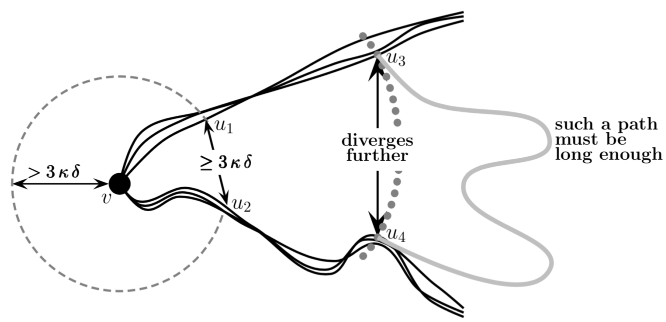


Figure 3: Illustration of Fact 2. Geodesic rays diverging sufficiently cannot connect back without using a sufficiently long path.

Fig. 3 pictorially illustrates this result.

# CHAPTER 3

# EFFECT OF GROMOV-HYPERBOLIC CURVATURE ON CUT AND EXPANSION

In this section we provide an upper bounds for node expansions for hyperbolic graph $G$. We show that many witnesses of subset of nodes satisfying such expansion bounds can be found efficiently in polynomial time while they form a nested (laminar) family, or have limited overlap in the sense that every subset has a certain number of "private" nodes not contained in any other subset.

These bounds also imply in an obvious manner corresponding upper bounds for the edge-expansion of $G$ and for the smallest non-zero eigenvalue of the Laplacian of $G$. To illustrate these non-trivial bounds, suppose that the maximum degree d and the hyperbolicity value $\delta$ grows asymptotically very slowly with respect to the number of nodes n, and the diameter $D$ to be of the order of the minimum possible value of $\log_d\ n$. In Remark 1, we provide an explanation of the asymptotic of these bounds in comparison to expander graphs. In particular, if $\delta$ is fixed (i.e., $G$ is hyperbolic) then d has to be increased to at least $2^{\Omega(\sqrt{\log\log n/\log\log\log n})}$ to get a positive non-zero Cheeger constant, whereas if $d$ is fixed then $\delta$ need to be at least $\Omega(\log n)$ to get a positive non-zero Cheeger constant (this last implication also follows from the results in).

In the last part we deal with the absolute size of $s-t$ cuts in hyperbolic graphs, and shows that a large family of s-t cuts having at most $d^{O(\delta)}$ cut-edges can be found in polynomial time in $\delta$-hyperbolic graphs when every node other than s and $t$ has a maximum degree of $d$ and the distance between $s$ and $t$ is at least $\Omega(\delta \log n)$.

### 3.1 Basic Notations and Assumptions

We first define basic concepts and terminologies throughout this chapter. We will simply write log to refer to logarithm base 2. Our basic input is an ordered triple $\langle G, d, \delta \rangle$ denoting the given *connected undirected unweighted* graph $G = (V, E)$ of hyperbolicity $\delta$ in which every node has a degree of at most $d > 2$. We will always use the variable m and n to denote the number of edges and the number of nodes, respectively, of the given input graph. In our analysis throughout the chapter, we assume that $n$ is always sufficiently large. For notational convenience, we will ignore floors and ceilings of fractional values in our theorems and proofs, e.g., we will simply write $n/3$ instead of $\lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$, since this will have no effect on the asymptotic nature of the bounds. *We will also make no serious effort to optimize the constants that appear in the bounds in our theorems and proofs.* In addition, the following notations will be used throughout the chapter:

- $|\mathcal{P}|$ is the length (number of edges) of a path $\mathcal{P}$ of a graph.
- $\overline{u, v}$ is a *shortest path* between nodes $u$ and $v$. In our proofs, any shortest path can be selected but, once selected, the *same* shortest path *must* be used in the remaining part of the analysis.
- $dist_H(u, v)$ is the distance (number of edges in a shortest path) between nodes $u$ and $v$ in a graph $H$ (and is $\infty$ if there is no path between $u$ and $v$ in $H$).

- $D(H) = max_{u,v \in V'}\{dist_H(u,v)\}$ is the diameter of the graph $H = (V', E')$. Thus, in particular, $u, v \in V$ for our input graph $G$ there exists two nodes $p$ and $q$ such that $dist_G(p,q) = D(G) \geq log_d n$.

- For a subset $S$ of nodes of the graph $H = (V', E')$, the *boundary* $\partial_H(S)$ of $S$ is the set of nodes in $V' \backslash S$ that are connected to at least one node in $S$, i.e.,

$$\partial_H(S) = \{u \in V' \backslash S | v \in S \wedge \{u,v\} \in E'\}$$

  Similarly, for any subset $S$ of nodes, $\text{cut}_H(S)$ denotes the set of edges of $H$ that have *exactly* one end-point in $S$.

  The readers should note that our definition of $\partial H(S)$ involved the set of the nodes, and **not the set of edges**, that are connected to $S$.

- $B_H(u,r)$ is the set of nodes contained in a ball of radius $r$ centered at node $u$ in a graph $H$, i.e., $B_H(u,r) = \{v | dist_H(u,v) \leq r\}$

## 3.2 Node and Edge Expansions

In this section we define the node (or edge) expansion ratios of a $\delta$-hyperbolic graph for some (not necessarily constant) $\delta$. The following definitions are standard in the graph theory literature and repeated here only for the sake of completeness.

**Definition 4** (**Node and edge expansion ratios of a graph**).

***(a)*** *The node expansion ratio $h_G(S)$ of a subset $S$ of at most $|V|/2$ nodes of a graph $G = (V, E)$*

*is defined as* $h_G(S) = \frac{|\partial G(S)|}{|S|}$*. If* $h_G(S) > c$ *for some constant* $c > 0$ *and for all subsets* $S$ *of at most* $|V|/2$ *nodes then we call* $G$ *a node-expander.*

**(*b*)** *The edge expansion ratio* $g_H(S)$ *of a subset* $S$ *of at most* $|V|/2$ *nodes of a graph* $G = (V, E)$ *is defined as* $g_G(S) = \frac{|cut_G(S)|}{|S|}$*. If* $h_G(S) > c$ *for some constant* $c > 0$ *and for all subsets* $S$ *of at most* $|S| \leq |V|/2$ *nodes then we call* $G$ *an edge-expander or sometimes simply an expander.*

**Definition 5 (Witness of node or edge expansions).** *A witness of a node (respectively, edge) expansion bound of* $c$ *of a graph* $G = (V, E)$ *is a subset* $S$ *of at most* $|V|/2$ *nodes of* $G$ *such that* $h_G(S) \leq c$ *(respectively,* $g_G(S) \leq c$*).*

**Notation** $\mathfrak{h}_G = \min_{S \subset V : |S| \leq |V|/2} \left\{ h_G(S) \right\}$ will denote the *minimum* node expansion of a graph $G = (V, E)$.

For any graph $G = (V, E)$, any subset $S$ containing exactly $|V|/2$ nodes has $|\partial_G(S)| \leq |V|/2$, and thus $0 < \mathfrak{h}_G = \min_{S \subset V : |S| \leq |V|/2} \left\{ h_G(S) \right\} \leq 1$ All our expansion bounds in this section will be stated for node expansions only. Since $g_G(S) \leq d\ hG(S)$ for any graph $G$ whose nodes have a maximum degree of $d$, our bounds for node expansions translate to some corresponding bounds for the edge expansions as well.

### 3.3 Overview of Our Results

Before proceeding with formal theorems and proofs, we first provide an informal non-technical intuitive overview of our results.

▷ Our first two results in Section 3 provide upper bounds for node expansions for the triple $\langle G, d, \delta \rangle$. as a function of $n, d$, and $\delta$. These two results, namely Theorem 6 and Theorem

8, provide absolute bounds and show that many witnesses (subset of nodes) satisfying such expansion bounds can be found efficiently in polynomial time satisfying two additional criteria:

- ▷ the witnesses (subsets) form a nested family, or
- ▷ the witnesses have limited overlap in the sense that every subset has a certain number of "private" nodes not contained in any other subset.

These bounds also imply in an obvious manner corresponding upper bounds for the edge-expansion of $G$ and for the smallest non-zero eigenvalue of the Laplacian of $G$. diverges further In Remark 1, we provide an explanation of the asymptotic of these bounds in comparison to expander-type graphs. For example, if $\delta$ is fixed (i.e., $G$ is hyperbolic) then $d$ has to be increased to at least $2^{\Omega\left(\sqrt{\log\log n/\log\log\log n}\right)}$ to get a positive non-zero Cheeger constant, whereas if $d$ is fixed then $\delta$ need to be at least $\Omega(\log n)$ to get a positive non-zero Cheeger constant (this last implication also follows from the results in (83; 103)).

▷ Our last result in this section, namely Lemma 9, deals with the absolute size of s-t cuts in hyperbolic graphs, and shows that a large family of s-t cuts having at most $d^{O(\delta)}$ cut-edges can be found in polynomial time in $\delta$-hyperbolic graphs when d is the maximum degree of any node except s, t and any node within a distance of $35\delta$ of s and the distance between s and t is at least $\Omega(\delta \log n)$. This result is later used in designing the approximation algorithm for minimizing bottleneck edges and small set expansion problems in Section 4 and 5 respectivley.

### 3.4 Effect of $\delta$ on Expansions and Cuts in $\delta$-hyperbolic Graphs

The two results in this section are related to the node (or edge) expansion ratios of a graph that is $\delta$-hyperbolic for some (not necessarily constant) $\delta$.

#### 3.4.1 Nested Family of Witnesses for Node/Edge Expansion

Here our goal is to find a large nested family of subsets of nodes with good node expansion bounds. A family of sets $S_1, S_2, ..., S_l$ is called *nested* if $S_1 \subset S_2 \subset ... \subset S_l$. For two nodes $p$ and $q$ of a graph $G = (V, E)$, a cut $S$ of $G$ that "separates $p$ from $q$" is a subset $S$ of nodes containing $p$ but not containing $q$, and the set of cut edges $cut_G(S, p, q)$ corresponding to the cut $S$ is the set of edges with exactly one end-point in $S$, *i.e.*,

$$cut_G(S,p,q) = \Big\{ \{u,v\} | p,u \in S \text{ and } q,v \in V \setminus S \Big\}$$

Recall that $d$ denotes the maximum degree of any node in the given graph $G$.

**Theorem 6.** *For any constant $0 < \mu < 1$, the following result holds for $< G, d, \delta >$. Let $p$ and $q$ be any two nodes of $G$ and let $\Delta = dist_G(p,q)$. Then, there exists at least $t = max\{\frac{\Delta^\mu}{56 \log d}, 1\}$ nested family of subsets of nodes $\emptyset \subset S_1 \subset S_2 \subset ... \subset S_l$ at each of at most $n/2$ nodes with the following properties:*

$\triangleright$ $\forall j \in \{1, 2, ..., t\}$ :

$$h_G(S_j) \leq \min\left\{ \frac{8 \ln(\frac{n}{2})}{\Delta}, \max\{ \frac{500 \ln n}{\Delta 2^{\frac{\Delta^\mu}{28\delta \log(2d)}}} \} \right\}.$$

▷ *All the subsets can be found in a total of* $O(n^3 \log n + mn^2)$ *time.*

▷ *Either all the subsets* $S_1, S_2, ..., S_t$ *contain the node* $p$*, or all of them contain the node* $q$*.*

**Corollary 7.** *Letting* $p$ *and* $q$ *be two nodes such that* $dist_G(p,q) = D(G) = D$ *realizes the diameter of the graph* $G$*, we get the bound:*

$$h_G(S_j) \leq \min\left\{\frac{8ln(\ln/2)}{D}, \max\left\{(\frac{1}{D})^{1-\mu}, (500\ln)/\left(D\ 2^{\frac{D^\mu}{28\ \delta\log(2d)}}\right)\right\}\right.$$

*Since* $D > \log n / \log d$*, the above bound implies:*

$$\mathfrak{h}_G(S_j) \leq \max\left\{(\log d/\log n)^{1-\mu}, (500\log d)/\left(2^{\log^\mu n/28\ \delta\log^{1+\mu}(2d)}\right)\right\} \tag{3.1}$$

**Remark 1.** *The following observations may help the reader to understand the asymptotic nature of the bound (3.1).*

***(a)*** *The first component of the bound is* $O(1/\log^{1-\mu} n)$ *for fixed* $d$*, and is* $\Omega(1)$ *only when* $d = \Omega(n)$*.*

***(b)*** *To better understand the second component of the bound, consider the following cases (recall that* $h_G = \Omega(1)$ *for an expander):*

- *Suppose that the given graph is a hyperbolic graph of constant maximum degree, i.e., both* $\delta$ *and* $d$ *are constants. In that case,*

$$(500\log d)/\left(2^{\frac{\log^{\mu} n}{28\ \delta\log^{1+\mu}(2d)}}\right) = O\left(1/\left(2^{O(1)\log^{\mu}\ n}\right)\right) = O(1/\ \textit{polylog}\ (n))$$

- *Suppose that the given graph is hyperbolic but the maximum degree $d$ is arbitrary. In that case,*

$$(500\log d)/\left(2^{\frac{\log^{\mu} n}{28\ \delta\log^{1+\mu}(2d)}}\right) = O(\log d/(2^{O(1)\log^{\mu}\ n/\log^{1+\mu} d})) = O(\log d/\textit{polylog}(n)^{1/\log^{1+\mu} d})$$

*and thus $d$ has to be increased to at least $2^{\Omega\left(\sqrt{\log\log n/\log\log\log n}\right)}$ to get a constant upper bound.*

- *Suppose that the given graph has a constant maximum degree but not necessarily hyperbolic (i.e., $\delta$ is arbitrary). In that case,*

$$(500\log d)/\left(2^{\frac{\log^{\mu} n}{28\ \delta\log^{1+\mu}(2d)}}\right) = O\left(1/2^{O(1)\log^{\mu} n/\delta}\right)$$

*and thus $\delta$ need to be at least $\Omega(\log^{\mu} n)$ to get a constant upper bound.*

#### 3.4.1.1 Proof of Theorem 6

To prove the bounds and consequently to show hyperbolic graphs are not expander, we use the same cylinder or ball removing approach in (83; 102). However, we try to find these witnesses while optimizing the corresponding expansion bounds. Our proof is divided into three steps. In step (I), we analyze the first component of the expansion bounds. In step (II), we analyze the second component, and in last step (step (III)) we show the time complexity of the algorithm.

(I) **Proof of the easy part of the bound,i.e.,** $h_G(S_j) \le (8\ln(n/2))/\Delta$

The proof for this part is straightforward. Without loss of generality, we assume $|B_G(p,\Delta)| \le \min\left\{|B_G(p,\Delta/2)|, |B_G(q,\Delta/2)|\right\} \le n/2$. Consider the sequence of balls $B_G(p,r)$ for $r = 0,1,2,...,\Delta/2$. Thus it follows that:

$$\begin{aligned} n/2 > |B_G(p,\Delta/2)| &\ge \prod_{l=0}^{(\Delta/2)-1} (1 + h_G(B_G(p,l))) \\ &\ge \prod_{l=0}^{(\Delta/2)-1} e^{h_G(B_G(p,l))/2} = e^{\sum_{l=0}^{(\Delta/2)-1}} h_G(B_G(p,l)) \\ \Rightarrow \ln(n/2) > &\sum_{l=0}^{(\Delta/2)-1} h_G(B_G(p,l))/2 \Rightarrow \frac{\sum_{l=0}^{(\Delta/2)-1} h_G(B_G(p,l))}{\Delta/2} < \frac{(4\ln\ (n/2))}{\Delta} \end{aligned}$$

By a simple averaging argument, there must now exist $\Delta/4 > \max\{\frac{\Delta^\mu}{56\log d}, 1\}$ distinct balls (subsets of nodes) $B_G(p,r_1) \subset B_G(p,r_2) \subset ... \subset B_G(p,r_{\Delta/4})$ such that $|B_G(p,r_j)| < (8\ln(n/2))/\Delta$ for $j = 1,2,...,\Delta/4$. It is straightforward to see that these balls can be found within the desired time complexity bound.

(II)**Proof of the difficult part of the bond,i.e.,** $h_G(S_j) \le \max\left\{(1/\Delta)^{1-\mu}, \frac{500\ln n}{\Delta 2^{\frac{\Delta^\mu}{28\ \delta\log(2d)}}}\right\}$

(II-a)**The easy case of** $\Delta = O(1)$

Here first we consider the case $\Delta = O(1)$ and then we will extend it to case $\Delta = \omega(1)$. When $\Delta = O(1)$, we know $\Delta = c$ for any some constant $c \ge 1$ then since $\delta \ge 1/2, d > 1$and n is sufficiently large, we have $(500\ln n)/(\Delta 2^{\Delta^\mu/(28\delta\log(2d))}) > (500\ln n)/(\Delta 2^{(1/14)\Delta^\mu}) > 1$. Thus, any subset of n/2 nodes containing p satisfies the claimed bound, and the number of such subsets is $\binom{\frac{n}{2}-1}{n-2} \gg t$.

(II-b) **The case of $\Delta = \omega(1)$**

For the case $\Delta = \omega(1)$ we assume that $D(n) = \omega(1)$, $lim_{n\to\infty} D(c) > c$ for any constant c. Let $p', q'$ be nodes on a shortest path between p and q such that $dist_G(p, p') = dist_G(p', q') = \Delta/3$. For our analysis we also set the value of parameter

$$\alpha = \alpha_0 = 1/(7\Delta^{1-\mu}\log(2d)) \tag{3.2}$$

where $\alpha_0$ is less than 1/4[1].

Let $C$ be set of nodes at a distance of $\lfloor \alpha\Delta \rfloor > \alpha\Delta - 1$ of a shortest path $\overline{p', q'}$ between $p'$ and $q'$. Thus,

$$C = \{u | \exists v \in \overline{p', q'} : dist_G(u, v) = \lceil \alpha\Delta \rceil\} \Rightarrow |C| \leq (\Delta/3)d^{\lfloor \alpha\Delta \rfloor} < (\Delta/3)d^{\alpha\Delta} \tag{3.3}$$

Let graph $G_{-C}$ is reached by removing the nodes in $C$ from graph $G$ and by Fact 1, we will have

$$dist_{G-C}(p, q) \geq (\Delta/60)2^{\alpha\Delta/\delta}. \tag{3.4}$$

---

[1]We will later need to vary value of $\alpha$ in our analysis.

Let $B_G(p,r)$ be the ball of radius r centered at node p in G with $|B_G(p,r)| \leq n/2$, and let $\bar{h}(p,j) := \left(\sum_{l=0}^{j-1} B_G(p,l)\right)/j$. Then, since $|B_G(p,0)| = 1$ and $\frac{B_G(p,r)}{B_G(p,r-1)} = 1 + h_G(B_G(p,r-1))$, we have

$$|B_G(p,r)| = \prod_{j=0}^{r-1}(1 + h_g(B_G(p,j)))$$

$$\geq \prod_{j=0}^{r-1} e^{h_G(B_G(p,j))/2} = e^{\sum_{j=0}^{r-1} h_G(B_G(p,j))/2} = e^{r\bar{h}(p,r)/2} \quad (3.5)$$

Assume without loss of generality that[1]

$$|B_{G-c}(p, dist_{G-c}(p,q)/2)| \leq |B_{G-c}(q, dist_{G-c}(p,q)/2)| \leq (n - |C|)/2 < \frac{n}{2} \quad (3.6)$$

**Case 1:** There exists a set of $t$ distinct indices $\{i_1, i_2, ..., i_t\} \subseteq \{0, 1, 2, ..., dist_{G_{-C}}(p\ , q)/2\}$ such that, $i_1 < i_2 < ... < i_t$ and, for all $1 \leq s \leq t$, $h_G(B_G(p, i_s)) = h_G(B_{G-c}(p, i_s)) \leq (1/\Delta)^{1-\mu}$ (see Fig.4). Then, the subsets $B_G(p, i_1) \subset B_G(p, i_2) \subset \cdots \subset B_G(p, i_t)$ satisfy our claim.

**Case 2:** If Case 1 does not hold. In this case, we have

$$\sum_{l=0}^{(\Delta/3)-\alpha\Delta-1} h_G(B_G(p,l)) > ((dist_{G-c}(p,q)/2) - (t-1))(1/\Delta)^{1-\mu}$$

$$> ((\Delta/3) - \alpha\Delta - t)(1/\Delta)^{1-\mu} > \Delta^{\mu/4} \quad (3.7)$$

---

[1]Note that if no path between nodes $p$ and $q$ in $G_{-C}$ then $dist_{G_{-C}}(p,q) = \infty$ and $B_{G_{-C}}\big(p, dist_{G_{-C}}(p,q)/2\big)$ and $B_{G_{-C}}\big(q, dist_{G_{-C}}(p,q)/2\big)$ contain all the nodes reachable from $p$ and $q$ respectively, in $G_{-C}$.

Let $r_p$ be the *least* integer such that $B_{G_{-C}}(p, r_p) = B_{G_{-C}}(p, r_p + 1)$. Since $G$ is a connected graph and, for all $r \leq (\Delta/3) - \alpha\Delta$ we have $B_G(p, r) \cap C = \emptyset \equiv B_{G_{-C}}(p, r) = B_G(p, r)$ we have $r_p \geq (\Delta/3) - \alpha\Delta$ (see Fig. 4).

The current strategy may be failed when $r_p$ is precisely $(\Delta/3) - \alpha\Delta$. This could be happen when $p$ is disconnected from $q$ in $G_{-C}$.

**Failure of the strategy in proof of Theorem 5**

Note that it is possible that $r_p$ is precisely $(\Delta/3) - \alpha\Delta$ or not too much above it (this could happen when p is disconnected from q in $G_{-C}$). Consequently, we may not be able to use our current technique of enlarging the ball $B_{G-C}(p, r)$ for r beyond $(\Delta/3) - \alpha\Delta$ to get the required number of subsets of nodes as claimed in the theorem. A further complication arises because, for $r > (\Delta/3) - \alpha\Delta$, expansion of the balls $B_G(p, r)$ in G-C may differ from that in G, i.e., $h_G(B_{G-C}(p, r))$ need not be the same as $h_{G-C}(B_{G-C}(p, r))$.

**Rectifying the current strategy**

We now change our strategy in the following manner. Let us write $r_p$ as $r_{p,\alpha\Delta}$ to show its dependence on $\alpha\Delta$ and let $\alpha_1 = \frac{1}{14\Delta^{1-\mu}\log(2d)}$. Vary $\alpha$ from $\alpha = \alpha_1$ to $\alpha = \alpha_1/2$ in steps of $-1/\Delta$, and consider the sequence of values $r_{p,\alpha_1\Delta}, r_{p,\alpha_1\Delta-1}, ..., r_{p,\alpha_1\Delta/2}$. Let $C_{\alpha_1\Delta-l}$ denote the set of nodes in C when $\alpha$ is set equal to $\alpha_1 - (l/\Delta)$ for $l = 0, 1, 2, ..., \alpha_1\Delta/2$ (see Fig. 6). Consider the two sets of nodes $C_{\alpha_1\Delta-l}$ and $C_{\alpha_1\Delta-l'}$ with $l < l'$. Obviously, $C_{\alpha_1\Delta-l} \neq C_{\alpha_1\Delta-l'}$ for any $l \neq l'$.

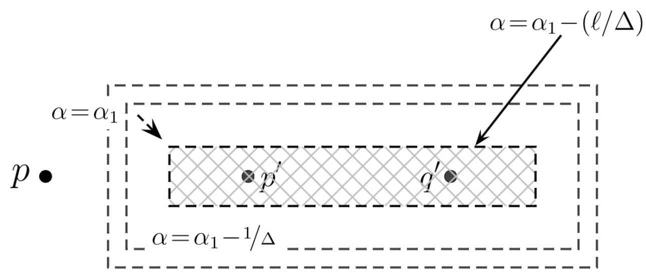


Figure 4: Illustration of various cases in the proof of Theorem 5. Nodes on the boundary of the lightly cross-hatched region belong to $C_{\alpha_1\Delta - l}$.

**Case 2.1 (relatively easier case):** Removal of each of the set of nodes $C_{\alpha_1\Delta}, C_{\alpha_1\Delta-1}, ..., C_{(\alpha_1\Delta)/2}$ disconnects p from q in the corresponding graphs G$-C_{\alpha_1\Delta}, G - C_{\alpha_1\Delta-1}, ..., G - C_{(\alpha_1\Delta)/2}$, respectively.

Then, for any $0 \leq l \leq (\alpha_1\Delta)/2$, we have

$$r_{p,\alpha_1\Delta-l} \geq (\Delta/3) - \alpha_1\Delta + l \geq (\Delta/3) - \alpha_1\Delta$$

$$\left|B_{G-C_{\alpha_1\Delta-l}}(p, r_{p,\alpha_1\Delta-l})\right| > \left|B_{G-C_{\alpha_1,\Delta}}\left(p, (\frac{\Delta}{3}) - \alpha_1\Delta\right)\right| \geq e^{\frac{1}{2}\sum_{j=0}^{(\Delta/3)-\alpha_1 D-1} h_G(B_G(p,j))}$$
$$> e^{\Delta^\mu/8}$$

$$\left|\partial_G\left(B_{G-C_{\alpha_1\Delta-l}}(p, r_{p,\alpha_1\Delta-l})\right)\right| \leq \left|C_{\alpha_1\Delta-l}\right| \leq \left|C_{\alpha_1\Delta}\right| < (\Delta/3)d^{\alpha_1\Delta}$$

$$h_G\left(B_{G-C_{\alpha_1\Delta-t}}(p, r_{p,\alpha_1\Delta-l})\right) = \frac{\left|\partial_G\left(B_{G-C_{\alpha_1\Delta-l}}(p, r_{p,\alpha_1\Delta-l})\right)\right|}{\left|B_{G-C_{\alpha_1\Delta-l}}(p, r_{p,\alpha_1\Delta-l})\right|}$$
$$< \frac{(\Delta/3)d^{\alpha_1\Delta}}{e^{\Delta^{\mu}/8}} = \frac{(\Delta/3)d^{\frac{\Delta^\mu}{14\log(2d)}}}{e^{\Delta^\mu/8}} < \frac{(\Delta/3)(d^{1/\log d})^{\Delta^\mu/14}}{e^{\Delta^\mu/8}}$$
$$= \frac{(\Delta/3)2^{\Delta^\mu/14}}{e^{\Delta^\mu/8}} < \frac{\Delta/3}{2^{\Delta^\mu/20}} < \left(\frac{1}{\Delta}\right)^{1-\mu}, \text{ since } \mu > 0 \text{ and } \Delta = \omega(1) \quad (3.8)$$

This inequality (3.8) implies that there exists a set of $1 + (\alpha_1\Delta)/2 = 1 + (\Delta^\mu)/(28\log(2d)) > \Delta^\mu/(56\log d)$

subset of nodes $B_{G-C_{\alpha_1\Delta}}(p, r_{p,\alpha_1\Delta}) \subset B_{G-C_{\alpha_1\Delta-1}}(p, r_{p,\alpha_1\Delta-1}) \subset \ldots \subset B_{G-C_{\alpha_1\Delta/2}}(p, r_{p,\alpha_1\Delta/2})$ such that each subset $B_{G-C_{\alpha_1\Delta-l}}(p, r_{p,\alpha_1\Delta-l})$ has $h_G(B_{G-C_{\alpha_1\Delta-l}}(p, r_p, \alpha_1\Delta - l)) < (1/\Delta)^{1-\mu}$. This proves our claim.

**Case 2.2 (the difficult case):** Case 2.1 does not hold.

This means that there exists an index $0 \leq t \leq (\alpha_1\Delta)/2$ such that the removal of the set of nodes in $C_{\alpha_1\Delta-t}$ does not disconnect p from q in the corresponding graphs $G_{-C\alpha_1\Delta-t}$. This implies $r_{p,\alpha_1\Delta-t} > dist_{G-C_{\alpha_1\Delta-t}(p,q)/2}$. For notational convenience, we will denote $C_{\alpha_{1\Delta}-t}$ and $G - C_{\alpha_1\Delta-t}$ simply by $C$ and $G_{-C}$, respectively. We redefine $\alpha_0 = \alpha_1 - (t/\Delta)$ such that $\alpha_1\Delta - t = \alpha_0\Delta$. Note that $\alpha_{1/2} \leq \alpha_0 \leq \alpha_1$.

**First goal: show that our selection of $\alpha_0$ ensures that removal of nodes in $C$ does**

**not decrease the expansion of the balls $B_{G_{-C}}(p,r)$ in the new graph $G_{-C}$ by more than a constant factor.**

First, note that the goal is trivially achieved if If $r \leq (\Delta/3) - \alpha_0\Delta$ since for all $r \leq (\Delta/3) - \alpha_0\Delta$ we have $h_{G_{-C}}(B_{G_{-C}}(p,r))$. Thus, assume that $r > (\Delta/3) - \alpha_0\Delta$. To satisfy our goal, it suffices if we can show the following assertion:

$$\forall (\Delta/3) - \alpha_0\Delta < r \leq dist_{G_{-C}}(p,q)/2 : h_G(B_{G_{-C}}(p,r-1)) > (1/\Delta)^{1-\mu} \Rightarrow$$
$$h_{G_{-C}}(B_{G_{-C}(p,r-1)}) \geq h_G(B_{G_{-C}}(p,r-1)/2) \quad (3.9)$$

We verify the above statement as shown below. First, note that:

$$\begin{aligned}
&h_{G-C}\big(B_{G-C}(p,r-1)\big) \geq h_G\big(B_{G-C}(p,r-1)\big)/2 \\
&\equiv \frac{\big|\partial_G(B_{G-C}(p,r-1))\big| - \big|\partial_G(B_{G-C}(p,r-1)) \cap C\big|}{\big|B_{G-C}(p,r-1)\big|} \geq h_G\big(B_{G-C}(p,r-1)\big)/2 \\
&\Leftarrow \frac{\big|\partial_G\big(B_{G-C}(p,r-1)\big)\big| - \big|C\big|}{\big|B_{G-C}(p,r-1)\big|} \geq h_G\big(B_{G-C}(p,r-1)\big)/2 \\
&\equiv \frac{\partial_G\big(B_{G-C}(p,r-1)\big)}{B_{G-C}(p,r-1)} - \frac{|C|}{B_{G-C}(p,r-1)} \geq h_G\big(B_{G-C}(p,r-1)\big)/2 \\
&\equiv h_G\big(B_{G-C}(p,r-1)\big) - \frac{\big|C\big|}{B_{G-C}(p,r-1)} \geq h_G\big(B_{G-C}(p,r-1)\Big)/2 \\
&\equiv \frac{2\big|C\big|}{h_G\big(B_{G-C}(p,r-1)\big)} \leq \big|B_{G-C}(p,r-1)\big| \\
&\Leftarrow \frac{2\big|C\big|}{h_G\big(B_{G-C}(p,r-1)\big)} \leq e^{\Delta^{\mu/8}},\ \text{since } \big|B_{G-C}(p,r-1)\big| \geq \left|B_{G-C}\left(p, (\frac{\Delta}{3}) - \alpha_0\Delta\right)\right|
\end{aligned}$$

$$= |B_{G-C}(p, (\frac{\Delta}{3}) - \alpha_0\Delta)| \geq |B_{G-C}(p, (\frac{\Delta}{3}) - \alpha_1\Delta)| \geq e^{\Delta^{\mu/8}}$$

$$\Leftarrow \big((\Delta/3)d^{\alpha_0\Delta})(2/(h_G(B_{G-C}(p, r-1)))\big) \leq e^{\Delta^{\mu/8}} since|C| < (\Delta/3)d^{\alpha_0\Delta}$$

$$\equiv (\Delta^{\mu}/8) \geq \ln\Delta + \alpha_0\Delta\ln d - \ln(3/2) - \ln\big(h_G(B_{G-C}(p, r-1))\big)$$

$$\Leftarrow (\Delta^{\mu}/8) \geq \ln\Delta + \alpha_0\Delta\ln d - \ln 3/2 - \ln\big(h_G(B_{G-C}(p, r-1))\big), since\alpha_0 \leq \alpha_1$$

$$\Leftarrow \alpha_1 \leq \frac{(\Delta^{\mu}/8) - \ln\Delta + \ln(h_G(B_{G-C}(p, r-1)))}{\Delta\ln d} \quad (3.10)$$

Now, if $h_G(B_{G-C}(p, r-1)) > (1/\Delta)^{1-\mu}$ then since $\Delta = \omega(1) we have:$

$(\Delta^{\mu}/8) - \ln(\Delta + \ln h_G(B_{G-C}(p, r-1))) > (\Delta^{\mu}/8) - \ln\Delta - (1-^{\mu})\ln\Delta > (\Delta^{\mu}/7)$

Thus, Inequality (3.10) is satisfied by our selection of $\alpha_1 = 1/(14\Delta^{1-\mu}\log(2d))$. This verifies (3.9) and satisfies our first goal.

**Second goal: Use the first goal and the fact that d$ist_{G-C}(p,q)$ is large enough to find the desired subsets.**

First assume that there exists a set of $t = max\{1, \Delta^{\mu}/(56\log d)\}$ indices $i_1 < i_2 < ... < i_t \in (\Delta/3) - \alpha_0\Delta + 1, (\Delta/3) - \alpha_0\Delta + 2, ..., (dist_{G-C}(p,q))/2$ such that

$$\forall 1 \leq s \leq t : h_G(B_{G-C}(p, i_s)) \leq (1/\Delta)^{1-\mu} \quad (3.11)$$

Obviously, the existence of these subsets $B_{G-C}(p, i_1) \subset B_{G-C}(p, i_2) \subset ... \subset B_{G-C}(p, i_t)$ proves our claim. Otherwise, there are no sets of t indices to satisfy last constraint. This implies that there exists a set of $\zeta = dist_G(p,q)/2 - ((\Delta/3) - \alpha_0\Delta)) - (t-1)$ distinct indices $j_1, j_2, ..., j_\zeta$ in $\{(\Delta/3) - \alpha_0\Delta + 1, (\Delta/3) - \alpha_0\Delta + 2, ..., (dist_{G-C}(p,q))/2\}$ such that

$$\forall 1 \le s \le \zeta : h_G(B_{G-C}(p, j_s)) > (1/\Delta)^{1-\mu} \Rightarrow$$
$$\forall 1 \le s \le \zeta : h_G - C(B_{G-C}(p, j_s)) \ge h_G(B_{G-C}(p, j_s))/2 \quad (3.12)$$

This in turn implies

$$\begin{aligned} \left|B_{G-C(p,dist_{G-C}(p,q))/2}\right| &> \left(\prod_{j=0}^{(\Delta/3)-\alpha_0-1} 1+h_G(B_{G-C}(p,j))\right)\left(\prod_{j=(\Delta/3)-\alpha_0}^{(\Delta/3)-\alpha_0+\zeta-1} (1+(h_G(B_{G-C}(p,j))/2))\right) \\ &> \left(\prod_{j=0}^{(\Delta/3)-\alpha_0\Delta-1} e^{h_G(B_{G-C}(p,j))/2}\right)\left(\prod_{(\Delta/3)-\alpha_0\Delta}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} e^{h_G(B_{G-C}(p,j))/4}\right) \\ &= \left(e^{\sum_{j=0}^{(\Delta/3)-\alpha_0\Delta-1} h_G(B_{G-C}(p,j))/2}\right)\left(e^{\sum_{(\Delta/3)-\alpha_0\Delta}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} h_G(B_{G-C}(p,j))/4}\right) > \\ & e^{\sum_{j=0}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} h_G(B_{G-C}(p,j))/4} \quad (3.13) \end{aligned}$$

Using (3.13) and our specific choice of the node $p$(over node $q$), we have

$$n/2 > \left|B_{G-C}(p, dist_{G-C}(p,q)/2)\right| > e^{\sum_{j=0}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} h_G(B_{G-C}(p,j))/4}$$
$$\Rightarrow \sum_{j=0}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} h_G\big(B_{G-C}(p,j)\big)/4 < 4\ln n \quad (3.14)$$

We know claim that there must exist a set of $t = \Delta^{\mu}/(56 \log d)$ distinct indices $i_1, i_2, ... < i_t$ in $\{0, 1, ..., (\Delta/3) - \alpha_0\Delta + \zeta - 1\}$ such that

$$\forall 1 \leq s \leq t : h_G(B_{G-C}(p, i_s)) \leq (500 \ln n)/\left(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))}\right) \tag{3.15}$$

The existence of these indices will obviously prove our claim. Suppose, for the sake of contradiction, that this is not the case. Together with (3.14) this implies:

$$\begin{aligned}
4 \ln n > & \sum_{j=0}^{(\Delta/3)-\alpha_0\Delta+\zeta-1} h_G(B_{G-C}(p, j)) \\
& > \left(\frac{\Delta}{3}) - \alpha_0\Delta + \zeta - (\Delta^{\mu}/(56 \log d)) + 1\right)\left((500 \ln n)/\left(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))}\right)\right) \\
\Rightarrow & \left((dist_{G-C}(p, q)/2) - \max\left\{1, (\Delta^{\mu}/(28 \log d))\right\}\right)\left((500 \ln n)/(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))})\right) \\
& < 4 \ln n, \\
\Rightarrow & \left((\Delta/120)2^{(\alpha_1\Delta)(2\delta)} - \max\left\{1, (\Delta^{\mu}/(28 \log d))\right\}\right)\left((500lnn)/(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))})\right) \\
& < 4 \ln n, \\
\equiv & \left(\left((\Delta/120)2^{(\Delta^{\mu}/\log(2d)}\right) - \max\left\{1, (\Delta^{\mu}/(28 \log d))\right\}\right)\left(125/\left(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))}\right)\right) \\
& < 1 \\
\Rightarrow & \left((\Delta/120)2^{(\Delta^{\mu}/\log(2d)}\right)\left(125/\left(\Delta 2^{\Delta^{\mu}/(28\delta \log(2d))}\right)\right) < 1 \equiv 125/121 < 1, \text{ since } \Delta = \omega(1)
\end{aligned} \tag{3.16}$$

Since (3.16) is false, there must exist a set of $t$ distinct indices $i_1 < i_2 < ... < i_t$ such that (3.15) holds and the corresponding sets $B_{G-C}(p, i_1) \subset B_{G-C}(p, i_2) \subset ... \subset B_{G-C}(p, i_t)$ prove our claim.

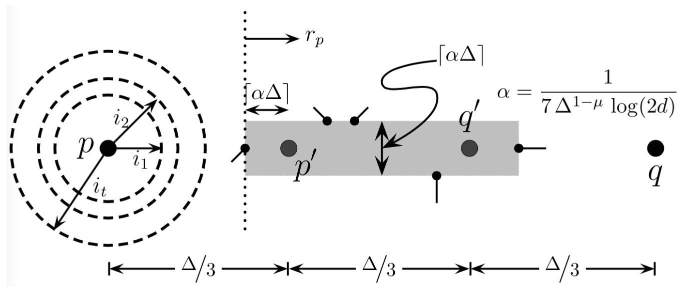


Figure 5: Illustration of various cases in the proof of Theorem 6. Nodes on the boundary of the lightly shaded region belong to $C_{\alpha_1}\Delta$.

(III)**Time complexity for finding each witness:**

Here we show the overview of the algorithm to find each witness provided and the whole time complexity. We can implement the following steps:

- Find two nodes p and q such that $dist_G(p, q) = \Delta$ in time $O(n^2 \log n + mn)$.
- Using breadth-first-search (BFS), find the two nodes $p', q'$ as in the proof in $O(m + n)$ time.
- There are at most $\alpha_1\Delta/2 = \Delta^\mu/(28\log(2d)) < n$ possible values of $\alpha$ considered in the proof. For each $\alpha$, the following steps are needed:
  - Use BFS to find the set of nodes C in $O(n^2 + mn)$ time.
  - Compute $G_{-C}$ in time $O(m + n)$.
  - Using BFS, compute $B_{G-C}(p, r)$ for every $0 \leq r \leq dist_{G-C}(p, q)/2$ in $O(m+n)$ time.

– Compute $h_G(B_{G-C}(p,r))$ for every $0 \leq r \leq dist_{G-C}(p,q)/2$ in $O(n^2+mn)$ time, and select a subset of nodes with a minimum expansion.

### 3.4.2 Family of Witnesses of Node/Edge Expansion With Limited Mutual Overlaps

In this section we will generate a family of cuts that are are sufficiently different from each other, i.e., they are either disjoint or have limited overlap. The following theorem addresses the central result.

**Theorem 8.** *Let $p$ and $q$ be any two nodes of $G$ with $dist_G(p,q) = \Delta > 8$. For any constant $0 < \mu < 1$ and any positive integer $\tau < \Delta/\left(\left(42\delta \log(2d)\log(2\Delta)\right)^{1/\mu}\right)$, we can find $\lfloor \tau/4 \rfloor$ distinct collections of subsets of nodes $\emptyset \subset \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{\lfloor \tau/4 \rfloor} \subset 2^V$ that each has at least $t_j = max\{\frac{(\Delta/\tau)^\mu}{56\log d}, 1\}$ nested family of subsets $\emptyset \subset V_{j,1} \subset V_{j,2}, ..., \subset V_{j,t_j} \subset V$ in time $O(n^3\log n + mn^2)$ such that:*

- $\forall j \in \{1, 2, ..., \lfloor \tau/4 \rfloor\}$ $\forall S \in \mathcal{F}_j$ :

$$h_G(S) \leq \max\left\{\left(\frac{1}{\Delta/\tau}\right)^{1-\mu}, \frac{360\log n}{\Delta/\tau 2^{\frac{(\Delta/\tau)^\mu}{7\delta\log(2d)}}}\right\}.$$

- *(limited overlap claim) For every pair of subsets $V_{i,k} \in \mathcal{F}_i$ and $V_{j,k'} \in \mathcal{F}_j$ with $i \neq j$, either $V_{i,k} \cap V_{j,k'} = \emptyset$ or at least $\Delta/(2\tau)$ nodes in each subset do not belong to the other subset.*

**Remark 2.** *Consider a bounded-degree hyperbolic graph, i.e., assume that $\delta$ and $d$ are constants. Setting $\tau = \Delta^{1/2}$ gives $\Omega(\Delta^{1/2})$ nested families of subsets of nodes, with each family having at* *least $\Omega(\Delta^{1/2})$ subsets each of maximum node expansion $(1/\Delta)^{(1-\mu)/2}$, such that every pairwise non- disjoint subsets from different families have at least $\Omega(\Delta^{1/2})$ private nodes.*

**P**roof. Select $\tau \leq \Delta/4$ such that $\tau$ satisfies the following:

$$\Delta/(60\tau)2^{((\Delta/\tau)^{\mu})/(28\delta\log(2d))} > (\Delta/\tau) + 2\Delta \tag{3.17}$$

Note that $\tau \geq (42\delta\log(2d)\log(2\Delta))^{1/\mu}/\Delta$ satisfies inequality (3.8) since

$$\Delta/(60\tau)2^{((\Delta/\tau)^{\mu})/(28\delta\log(2d))} > (\Delta/\tau) + 2\Delta$$

$$\Rightarrow (\Delta/\tau)^{\mu} > 28\delta\log(2d)\log(60+120\tau) > 168\delta\log(2d)\log(2\Delta)$$

$$\Rightarrow \tau < \Delta/((42\delta\log(2d)\log(2\Delta))^{1/\mu}), \text{ since } \tau < \Delta/4$$

Let $(p = p_1, p_2, ..., p_{\tau+1} = q)$ be an ordered sequence of $\tau+1$ nodes such that $dist_G(p_i, p_{i+1}) = \Delta/\tau$ for $i = 1, 2, ..., \tau$. Applying Theorem 5 for each pair $(p_i, p_{i+1})$, we get a nested family $\emptyset \subset \mathcal{F}_i \subset 2^V$ of subsets of nodes such that $t_i = |\mathcal{F}_i| \geq \max\{\frac{(\Delta/\tau)^{\mu}}{56\log d}, 1\}$ , and, for any $V_{i,k} \in \mathcal{F}_i$, $h_G(V_i, k) \leq \max\left\{(1/(\Delta/\tau))^{1-\mu}, (360\log n)/((\Delta/\tau)2^{(\Delta/\tau)^{\mu}/(7\delta\log(2d))})\right\}$

Recall that the subset of nodes $V_{i,k}$ was constructed in Theorem 5 in the following manner (see Fig. 5 for an illustration):

- Let $l_i$ and $r_i$ be two nodes on a shortest path $\overline{p_i, p_{i+1}}$ such that $dist_G(p_i, l_i) = dist_G(l_i, r_i) = dist_G(r_i, p_{i+1}) = dist_G(p_i, p_{i+1})/3$.

- For some $1/\big(28(\Delta/\tau)^{1-\mu}\log(2d)\big) \le \alpha_{i,k} \le 1/(14(\Delta/\tau)^{1-\mu}\log(2d)\big) < 1/4$, construct the graph $G_{-C_{i,k}}$ obtained by removing the set of nodes $C_{i,k}$ which are exactly at a distance of $\lceil\ \alpha_{i,k} dist_G(p_i, p_{i+1})\rceil$ from some node of the shortest path $\overline{l_i, r_i}$.

- The subset $V_{i,k}$ is then the ball $B_{G_{-C_{i,k}}}(y_i, a_{i,k})$ for some $a_{i,k} \in [0,\ dist_{G_{-C_i,k}}(p_i, p_(i+1))/2]$ and for some $y_i \in \{p_i, p_{i+1}\}$. If $y_i = p_i$ then we call the collection of subsets $\mathcal{F}_i$ "left handed", otherwise we call $\mathcal{F}_i$ "right handed".

We can partition the set of $\tau$ collections $\mathcal{F}_1, ..., \mathcal{F}_\tau$ into four groups depending on whether the subscript j of $\mathcal{F}_j$ is odd or even, and whether $\mathcal{F}_j$ is left handed or right handed. One of these 4 groups must at least $\lfloor \tau/4 \rfloor$ family of subsets. Suppose, without loss of generality, that this happens for the collection of families that contains $\mathcal{F}_{i,k}$ when i is even and $\mathcal{F}_{i,k}$ is left handed (the other cases are similar). We now show that subsets in this collection that belong to different families do satisfy the limited overlap claim.

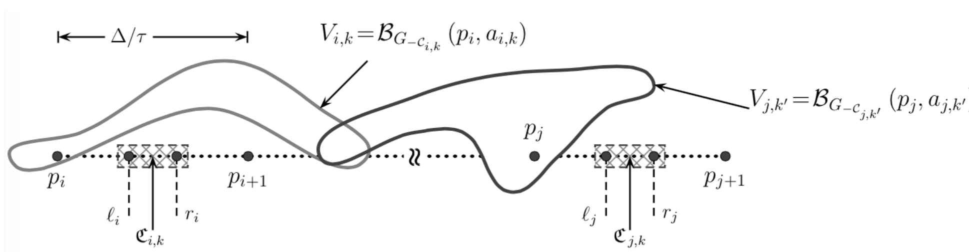


Figure 6: Illustration of various quantities related to the proof of Theorem 6. Nodes within the lightly cross-hatched region belong to $\mathfrak{C}_{i,k}$ and $\mathfrak{C}_{j,k'}$. Note that $B_{G_{-C_{i,k}}}(p_i, a_{i,k})$ and $B_{G_{-C_{j,k'}}}(p_j, a_{j,k'})$ need not to be balls in the original graph $G$.

Consider an arbitrary set in the above-mentioned collection of the form $V_{i,k} = B_{G-C_{i,k}}(p_i, a_{i,k})$ with even i. Let $\mathfrak{C}_{i,k}$ denote the nodes in the interior of the closed cylinder of nodes in G which are at a distance of at most $\lceil \alpha_{i,k} dist_G(p_i, p_{i+1}) \rceil$ from some node of the shortest path $\overline{l_i, r_i}$, i.e., let $\mathfrak{C}_{i,k} = \{u | \exists v \in \overline{l_i, r_i} : dist_G(u, v) \leq \lceil \alpha_{i,k} dist_G(p_i, p_{i+1}) \rceil\}$(see Fig. 3). Let $V_j, k' = B_{G-C_{j,k'}}(p_j, a_{j,k'})$ be a set in another family $\mathcal{F}_j$ with even $j \neq i$ (see Fig. 3). Assume, without loss of generality, that i is smaller than j, i.e., $i \leq j - 2$ (the other case is similar).

**Proposition 1.** $\mathfrak{C}_{i,k} \cap B_{G-c_{j,k'}}(p_j, \Delta/(2\tau)) = \phi$

**P**roof. Assume for the sake of contradiction that $u \in \mathfrak{C}_{i,k} \cap B_{G-C_{j,k'}}(p_j, \Delta/(2\tau)) \neq \phi$. Since $u \in \mathfrak{C}_{i,k}$, there exists a node $v \in \overline{l_i, r_i}$ such that $dist_G(v, u) \leq \lceil \alpha_{i,k} dist(p_i, p_{i+1}) \rceil < dist_G(p_i, p_{i+1})/4 = \Delta/(4\tau)$.

Thus,

$$u \in B_{G-C_{j,k'}}(p_j, \Delta/(2\tau)) \Rightarrow dist_{G-C_{j,k'}}(u, p_i) \leq \Delta/(2\tau) \Rightarrow dist_G(u, p_j) \leq$$
$$\Delta(2\tau) \Rightarrow dist_G(v, p_j) \leq dist_G(u, v) + dist_G(u, p_j) < \Delta/(4\tau) + \Delta/(2\tau) < \Delta/\tau$$

which contradicts the fact that $dist_G(v, p_j) > dist_G(v, p_j) > dist_G(p_{i+1}, p_j) = \Delta/\tau$. ❑

**Proposition 2.** *$dist_{G-C_{j,k'}} > \Delta/(2\tau)$ for any node $u \in V_{i,k} \cap V_{j,k'} = B_{G-C_{i,k}}(p_i, a_{i,k}) \cap B_{G-C_{j,k'}}(p_j, a_{j,k'})$.*

**P**roof. Assume for the sake of contradiction that $z = dist_{G-C_{j,k'}}(u, p_i) \leq \Delta/(2\tau)$. Since $u \in V_{i,k} = B_{G-C_{i,k}}(p_i, a_{i,k})$, this implies $dist_{G-C_{i,k}}(p_{i,k}, u) \leq a_{i,k} \leq dist_{G-C_{i,k}}/2$

Since $u \in V_{j,k'} = B_{G-C_{i,k}}(p_j, a_{j,k'})$ for some $a_{j,k'}$, this implies $u \in B_{G-C_{j,k'}}(p_j, z)$. Since $z \leq \Delta/(2\tau)$, by Proposition 1 $\mathfrak{C}_{i,k} \cap B_{G-C_{j,k'}} = \phi$, and therefore

$\Delta/(2\tau) \geq z = dist_{G-C_{j,k'}}(u, p_j) = dist_{G-C_{i,k} \cup C_{j,k'}}(u, p_j) \geq dist_{G-C_{i,k}}(u, p_j)$

since $\mathfrak{C}_{i,k} \cap B_{G-C_{j,k'}}(p_j, z) = \phi$ which in turn implies

$$dist_{G-C_{i,k}}(p_i, p_j) \leq dist_{G-C_{i,k}}(p_i, u) + dist_{G-C_{i,k}}(u, p_j) \leq dist_{G-C_{i,k}}(p_i, p_{i+1}) /2 + \Delta/(2\tau) \quad (3.18)$$

Since Hausdorff distance between the two shortest paths $\overline{l_i, r_i}$ and $\overline{p_j, p_{j+1}}$ is at least $(j-i-1)\Delta/\tau + \Delta/(3\tau) > \alpha_{i,k} dist_G(p_i, p_{i+1})$ and $dist_{G-C_{i,k}}(p_j, p_{i+1})$

$= (j-i)\Delta/\tau < \Delta$, we have

$$dist_{G-C_{i,k}}(p_i, p_{i+1})) \leq dist_{G-C_{i,k}}(p_j, p_{i+1}) \leq dist_{G-C_{i,k}}/2 + \Delta/(2\tau) + \Delta$$
$$\Rightarrow dist_{G-C_{i,k}}(p_i, p_{i+1}) \leq \Delta/\tau + 2\Delta \quad (3.19)$$

On the other hand, by Fact 1we have:

$$dist_{G-C_i}(p_i, p_{i+1}) \geq \Delta/(60\tau)2^{(\alpha_{i,k\Delta})/(\delta\tau)} \geq \Delta/(60\tau)2^{((\Delta/\tau)^{\mu})/(28\delta \log(2d))} \quad (3.20)$$

Inequalities (3.19) and (3.20) together imply

$$\Delta/(60\tau)2^{((\Delta/\tau)^{\mu})/(28\delta log(2d))} \leq (\Delta/\tau) + 2\Delta \quad (3.21)$$

Inequality (3.21) contradicts Inequality (3.17). ❑

To complete the proof of limited overlap claim, suppose that $V_{i,k} \cap V_{j,k'} \neq \phi$ and let $u \in V_{i,j} \cap V_{j,k'}$. Proposition 2 implies that $V_{j,k'} \supset B_{G-C_{j,k'}}(p_j, \Delta/(2\tau))$,
$u \notin B_{G-C_{j,k'}}(p_j, \Delta/(2\tau))$, and thus there are t least $\Delta/(2\tau)$ node on a shortest path in $G - C_{j,k'}$ from $p_j$ to a node at a distance of $\Delta/(2\tau)$ from $p_j$ that are not in $V_{i,k}$. ❑

### 3.4.3 Family of Mutually Disjoint Cuts

Given two distinct nodes $s, t \in V$ of a graph $G = (V, E)$ and a cut that separates s from t, we have following definitions:

- The cut-edges $\xi_G(S, s, t)$ corresponding to this cut is the set of edges with one end-point in S,

  $\xi_G(S, s, t) = \{\{u, v\} | u \in S, v \in V \backslash S, \{u, v\} \in E\}$,

- Cut-nodes $\mathcal{V}_G(S, s, t)$ corresponding to this cut is the end-points of these cut-edges that belong to S,

  $\mathcal{V}_G(S, s, t) = \{u | u \in S, v \in V \backslash S, \{u, v\} \in E\}$

**Note that in the following lemma $d$ is the maximum degree of any node "except $s, t$ and any node within a distance of $35\delta$ of s"** (degrees of these nodes may be arbitrary).

**Lemma 9.** *Given $\langle G, d, \delta \rangle$, where $d$ is the maximum degree of any node except s, t and any node within a distance of $35\delta$ of s. Suppose s and t are two nodes of G such that $dist_G(s, t) > 48\delta + 8\delta \log n$. There exists a set of at least $(dist_G(s, t) - 8\delta \log n)/(50\delta) = \Omega(dist_G(s, t))$ (node and edge) disjoint cuts such that each such cut has at most $d^{12\delta+1}$ cut edges.*

**Remark 3.** *Suppose that G is hyperbolic ( i.e., $\delta$ is a constant), d is a constant, and s and t be two nodes such that* $dist_G(s,t) > 48\delta + 8\delta \log n = \Omega(\log n)$*. Lemma 9 then implies that there are* $\Omega(dist_G(s,t))$ *s-t cuts each having O(1) edges. If, on the other hand,* $\delta = O(\log \log n)$*, then such cuts have polylog (n) edges.*

**Remark 4.** *The bound in Lemma 9 is obviously meaningful only if* $\delta = o(\log n)$*. If* $\delta = \Omega(logn)$*, then* $\delta$*-hyperbolic graphs include expanders and thus many small-size cuts may not exist in general.*

**P**roof. Recall that we may assume that $\delta \geq 1/2$. We start by doing a BFS starting from node s. Let $\mathcal{L}_i$ be the sets of nodes at the $i^{th}$ level (i.e., $\forall u \in \mathcal{L}_i : dist_G(s,u) = i$); obviously $t \in \mathcal{L}_{dist_G(s,t)}$. Assume $dist_G(s,t) > 48\delta + 8\delta logn$, and consider two arbitrary paths $P_1$ and $P_2$ between s and t passing through two nodes $v_1, v_2 \in \mathcal{L}_j$ for some $48\delta \leq j dist_G(s,t) - 7\delta \log n$.

We first claim that $dist_G(v_1, v_2) < 12\delta$. Suppose, for the sake of contradiction, suppose that $dist_G(v_1, v_2) \geq 12\delta$. Let $v_1'$ and $v_2'$ be the first node in level $\mathcal{L}_{j+6\delta \log n}$ visited by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Since both $\mathcal{P}_1$ and $\mathcal{P}_2$ are paths between s and t and $j + 6\delta \log n < dist_G(s,t)$ implies $\mathcal{L}_{j+6\delta \log n+1} \neq \emptyset$, there must be a path $\mathcal{P}_3$ between $v_1'$ and $v_2'$ through t using nodes not in $\bigcup_{0 \leq l \leq j+6\delta \log n \mathcal{L}_l}$. We show that this is impossible by Fact 2. Set the parameters in Fact 2 in the following manner: $\kappa = 4, \alpha = 6\delta \log n, r = j > 12\kappa\delta = 48\delta; u_1 = v_1, u_2 = v_2, u_4 = v_1'$,and $u_3 = v_2'$ . Then the length of $\mathcal{P}_3$ satisfies $|\mathcal{P}_3| > 2^{\log n + 5} > n$ which is impossible since $|\mathcal{P}_3| < n$.

We next claim that, for any arbitrary node in level $v \in \mathcal{L}_j$ lying on a path between s and t, $B_G(v, 12\delta)$ provides an s-t cut $cut_G(B_G(v,12\delta), s, t)$ having at most $\xi_G(B_G(v,12\delta), s, t) \leq d^{12\delta+1}$ edges. To see this, consider any path P between s and t and let u be the first node in $L_j$ visited by the path. Then, $dist_G(u, v) \leq 12\delta$ and thus $v \in B_G(v, 12\delta)$. Since nodes in $B_G(v, 12\delta)$ are at a distance of at least $35\delta$ from s and $t \in /B_G(v, 12\delta)$, d is the maximum degree of any node in $B_G(v, 12\delta)$ and it follows that $\xi_G(B_G(v, 12\delta), s, t) \leq d\partial G(B_G(v, 12\delta - 1)) \leq d^{12\delta+1}$.

We can now finish the proof of our lemma in the following way. Assume that $dist_G(s, t) > 48\delta + 8\delta \log n$. Consider the levels $\mathcal{L}_j$ for $j \in \{50\delta, 100\delta, 150\delta, ..., (dist_G\ (s, t) - 8\delta \log n)/(50\delta)\}$. For each such level $L_j$, select a node $v_j$ that is on a path between s and t and consider the subset of edges $cut_G(B_G(v_j, 12\delta), s, t)$. Then, $cut_G(B_G(v_j, 12\delta), s, t)$ over all j provides our family of s-t cuts. The number of such cuts is at least $(dist_G(s, t) - 8\delta logn)/(50\delta)$. To see why these cuts are node and edge disjoint, note that $\xi_G(B_G(v_j, 12\delta), s, t) \cap \xi_G(B_G(v_l, 12\delta), s, t) = \phi$ and $V_G(B_G(v_j, 12\delta), s, t) \cap V_G(B_G(v_l, 12\delta), s, t) = \phi$ for any $j \neq l$ since $dist_G(v_j, v_l) > 50\delta$. ❑

# CHAPTER 4

# HYPERBOLICITY AND NETWORK DESIGN APPLICATION

In this section, we consider a few algorithmic applications of the bounds and proof techniques we showed in the previous section. we discuss some applications of these bounds in designing improved approximation algorithms for two graph-theoretic problems for $\delta$-hyperbolic graphs when $\delta$ does not grow too fast as a function of $n$: We show in Section 4.1 (Lemma 10) that the problem of identifying vulnerable edges in network designs by minimizing shared edges ad- mits an improved approximation provided $\delta = o(\log n / \log d)$. We do so by relating it to a hitting set problem for size-constrained cuts (Lemma 11) and providing an improved approximation for this latter problem (Lemma 12). We also observe that obvious greedy strategies fail for such problems miserably.

## 4.1 Minimizing Bottleneck Edges Problem

In this section we consider the following problem.

**Problem 1 (Unweighted Uncapacitated Minimum Vulnerability problem (Uumv) (81; 106; 115))** The input to this problem a graph $G = (V, E)$, two nodes $s, t \in V$ , and two positive integers $0 < r < \kappa$. The goal is to find a set of $\kappa$ paths between s and t that minimizes the number of "shared edges", where an edge is called shared if it is in more than $r$ of these $\kappa$ paths between $s$ and $t$. When $r = 1$, the Uumv problem is called the "minimum shared edges" (Mse) problem.

We will use the notation $OPT_{Uumv}(G, s, t, r, \kappa)$ to denote the number of shared edges in an optimal solution of an instance of Uumv. Uumv has applications in several communication network design problems (see (113; 115) for further details). The following computational complexity results are known regarding Uumv and Mse for a graph with $n$ nodes and $m$ edges (see (81; 106)):

- Mse does not admit a $2^{log^{1-\epsilon} n}$-approximation for any constant $\epsilon > 0$ unless $NP \in DTIME(n^{\log \log n})$.
- Uumv admits a $\lfloor \kappa/(r+1) \rfloor$-approximation. However, no non-trivial approximation of Uumv that depends on m and/or n only is currently known.
- Mse admits a min $\{n^{3/4}, m^{1/2}\}$-approximation.

### 4.1.1 Greedy Fails for Uumv or Mse Even for Hyperbolic Graphs (i.e., Graphs With Constant $\delta$)

Several routing problems have been looked at for hyperbolic graphs (i.e., constant $\delta$) in the literature before (e.g., see (92; 101)) and, for these problems, it is often seen that simple greedy strategies do work. However, that is unfortunately not the case with Uumv or Mse. For example, one obvious greedy strategy that can be designed is as follows.

**(*Greedy strategy*)**

**Repeat** $\kappa$ times

Select a new path between s and t that shares a minimum number of edges with the already selected paths

The above greedy strategy can be arbitrarily bad even when $r = 1, \delta \leq 5/2$ and every node except s and t has degree at most three as illustrated in Fig. 7; even qualifying the greedy step by selecting a shortest path among those that increase the number of shared edges the least does not lead to a better solution.

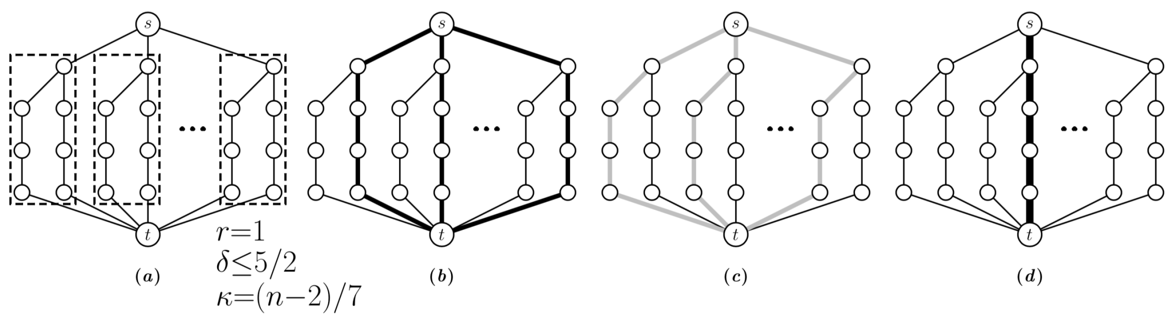


Figure 7: A bad example for the obvious greedy strategy. (a) The given graph in which every node except s and t has degree at most 3 and $\delta \leq 5/2$ . (b) Greedy first selects the $(n-2)/14$ edge-disjoint shortest paths shown in thick black. (c) Greedy then selects the shortest paths shown in light gray one by one, each of which increases the number of shared edges by one more. Thus, greedy uses $(n-2)/7$ shared edges. (d ) An optimal solution uses only 5 edges, i.e., $OPT_{Uumv}(G, s, t, 1, \kappa) = 5$.

### 4.1.2 Improved Approximations for Uumv or Mse for $\delta$ Up To $o(\frac{\log n}{\log d})$

**Note that in the following lemma d is the maximum degree of any node "except $s$, $t$ and any node within a distance of $35\delta$ of s"** (degrees of these nodes may be arbitrary). For up to $\delta = o(\log n/\log d)$, the lemma provides the first non-trivial approximation of Uumv as a function of n only (independent of $\kappa$) and improves upon the currently best min $\{n^{3/4}, m^{1/2}\}$-approximation of Mse for arbitrary graphs.

**Lemma 10.** *Let d be the maximum degree of any node except $s$, $t$ and any node within a distance of $35\delta$ of $s$ (degrees of these nodes may be arbitrary). Then, Uumv (and, consequently also Mse) for a $\delta$-hyperbolic graph $G$ can be approximated within a factor of $O\big(\max\{\log n, d^{O(\delta)}\}\big)$.*

**Remark 5.** *Thus for fixed d Lemma 2 provides improved approximation as long as $\delta = o(\log n)$.* ***Note that our approximation ratio is independent of the value of $\kappa$.*** *Also note that $\delta = \Omega(\log n)$ allows expander graphs as a sub-class of $\delta$-hyperbolic graphs for which Uumv or Mse is expected to be harder to approximate.*

Proof of Lemma 2

Our proof strategy has the following two steps:

- We define a new more general problem which we call the edge hitting set problem for size constrained cuts (Ehssc), and show that Uumv (and thus Mse) has the same approximability properties as Ehssc by characterizing optimal solutions of Uumv in terms of optimal solutions of Ehssc.

- We then provide a suitable approximation algorithm for Ehssc.

#### 4.1.2.1 Edge hitting set for size-constrained cuts (Ehssc)

The input to Ehssc is a graph $G = (V, E)$, two nodes $s, t \in V$, and a positive integer $0 < k \leq |E|$. Define a size-constrained s-t cut to be a s-t cut S such that the number of cut-edges cutG(S,s,t) is at most k. The goal of Ehssc is to find a hitting set of minimum cardinality for all size-constrained s-t cuts of G, i.e., find $\tilde{E} \subset E$ such that $|\tilde{E}|$ is minimum and

$$\forall s \in S \subset V \backslash \{t\} : |\xi_G(S, s, t)| \leq k \Rightarrow \xi_G(S, s, t) \cap \tilde{E} \neq \emptyset$$

We will use the notation $E_{Ehssc}(G, s, t, k)$ to denote an optimal solution containing OPTEhssc (G, s, t, k) edges of an instance of Ehssc.

**Lemma 11** (Relating Ehssc to Uumv). $OPT_{Uumv}(G, s, t, r, \kappa) = OPT_{Ehssc}(G, s, t, \lceil \kappa/r \rceil - 1)$.

**P**roof. Note that any feasible solution for Uumv must contain at least one edge from every collection of cut-edges EG(S,s,t) satisfying $|\xi_G(S, s, t)| \leq \kappa/r - 1$, since otherwise the number of paths going from $\xi_G(S, s, t) to V \backslash \xi_G(S, s, t)$ is at most $r\kappa 1 < \kappa$. Thus we get $OPT_{Uumv}(G, s, t, r, ) \geq OPT_{EHSSC}(G, s, t, \lceil \kappa \backslash r \rceil - 1, )$

On the other hand, $OPT_{Uumv}(G, s, t, r, \kappa) \leq OPT_{Ehssc}(G, s, t, \lceil/r\rceil 1)$ can be argued as follows. Consider the set of edges $\xi_{Ehssc}(G, s, t, \lceil\kappa/r\rceil 1)$ in an optimal hitting set, and set the capacity c(e) of every edge e of G as

$$c(e) = \begin{cases} \infty & \text{, if } \ e \in E_{Hssc}(G, s, t, \lceil\kappa/r\rceil) \\ \\ r & \text{, otherwise} \end{cases}$$

The value of the minimum s-t cut for G is then at least $\min\{\infty, r\lceil\kappa\backslash r\rceil\} \geq \kappa$ which implies (by the standard max-flow-min-cut theorem) the existence of $\kappa$ flows each of unit value. The paths taken by these $\kappa$ flows provide our de- sired $\kappa$ paths for Uumv. Note that at most r paths go through any edge e with $c(e)\backslash = \infty$ and thus $OPTUumv(G, s, t, r, \kappa) \leq e|c(e)\backslash = \infty = OPT_{EHSSC}(G, s, t, \lceil\kappa/r\rceil - 1)$ ❑

Now, we turn to providing a suitable approximation algorithm for Ehssc. Of course, Ehssc has the following obvious exponential-size LP-relaxation since it is after all a hitting set problem:

$$minimize \sum_{e \in E} x_e \ \text{ subject to}$$

$$\forall s \in S \subset V\backslash\{t\} \text{ such that } cut_G(S, s, t) \leq k : \sum_{e \in \xi_G(S,s,t)} x_e \geq 1$$

$$\forall e \in E : x_e \geq 0$$

Intuitively, there are at least two reasons why such a LP-relaxation may not be of sufficient interest. Firstly, known results may imply a large integrality gap. Secondly, it is even not very clear if the LP-relaxation can be solved exactly in a time efficient manner. Instead, we will exploit the hyperbolicity property and use Lemma 9 to derive our approximation algorithm.

**Lemma 12.** ***[Approximation algorithm for EHSSC]*** *EHSSC admits a* $O\big(\max\big\{\delta\log n, d^{o(\delta)}\big\}\big)$*-approximation.*

**P**roof. Our algorithm for Ehssc can be summarized as follows:

---

Algorithm for Ehssc

---

**If** $k \le d^{12\delta+1}$ **then**
  $\mathcal{A} \leftarrow \emptyset$, $j \leftarrow 0$, set the capacity $c(e)$ of every edge $e$ to 1
  **while** there exists a $s$-$t$ cut of capacity at most $k$ **do**
    $j \leftarrow j+1$, let $\mathcal{F}_j$ be the edges of a $s$-$t$ cut of capacity at most $k$
    $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{F}_j$, set $c(e) = \infty$ for every edge $e \in \mathcal{F}_j$
  **return** $\mathcal{A}$ as the solution
**else** $(*\ k > d^{12\delta+1}\ *)$
  **return** all the edges in a shortest path between $s$ and $t$ as the solution $\mathcal{A}$

---

The following case analysis of the algorithm shows the desired approximation bound.

**Case 1:** $k \le d^{12\delta+1}$. Let $\mathcal{F}_1, \mathcal{F}_2, ...\mathcal{F}_l$ be the sets whose edges were added to $\mathcal{A}$; thus, $|\mathcal{A}| \le kl$. Since $|\mathcal{F}_j| \le k$ and $\mathcal{F}_j \cap \mathcal{F}_{j'} = \emptyset$ for $j \ne j'$, $\mathbf{OPT}_{EHSSC}(G,s,t,k) \ge l$, thus providing an approximation bound of $k \le d^{12\delta+1}$.

**Case 2:** $k \le d^{12\delta+1}$ and $dist_G(s,t) \le 48\delta + 8\delta\log n$. Since $\mathbf{OPT}_{EHSSC}(G,s,t,k) \ge 1$, this provides a $O(\delta\log n)$-approximation.

**Case 3:** $k > d^{12\delta+1}$ and $dist_G(s,t) > 48\delta + 8\delta\log n$. Use Lemma 1 to find a collections $S_1, S_2, ..., S_l$ of $l = dist_G(s,t) - 8\delta\log n)/(50\delta)$ edge and node disjoint s-t cuts. Since $\text{cut}_G(S_{,j}\,, s, t) \le$

$d^{12\delta+1} < k$, any valid solution of EHSSC must select at least pne edge from $\xi_G(S_j, s, t)$. Since the cuts are edge and node disjoint, it follows that

$$OPT_{EHSSC}(G, s, t, k) \geq (dist_G(s, t) - 8\delta \log n)/(50\delta).$$

Since we return all the edges in a shortest path between a and t as the solution, approximation ratio achieved is $dist_G(s,t)/\left(\frac{dist_G(s,t)-8\delta \log n}{50\delta} < 100\delta\right)$ ❑

# CHAPTER 5

# HYPERBOLICITY AND SMALL-SET EXPANSION PROBLEM

In this section, we provide a polynomial-time solution for a type of small-set expansion problem originally proposed by Arora, Barak and Steurer (80) for the case when $\delta$ is sub-logarithmic in $n$.

## 5.1 Small-set Expansion Problem

The small set expansion (Sse) problem was studied by Arora, Barak and Steurer in(80) (and also by several other researchers such as ) in an attempt to understand the computational difficulties surrounding the Unique Games Conjecture (UGC). To define Sse, we will also use the normalized edge-expansion of a graph which is defined as follows (91). For a subset of nodes S of a graph G, let volG(S) denote the sum of degrees of the nodes in $G$. Then, the normalized edge expansion ratio $\phi_G(S)$ of a subset S of nodes of at most $|V|/2$ nodes of G is defined as $G(S)$ =$\mathrm{cut}_G(S)/volG(S)$. Since we will deal with only d-regular graphs in this subsection, $G(S)$ will simplify to $\mathrm{cut}_G(S)/(d|S|)$.

**Definition 13** (**(Sse Problem) [ a case of [(80), Theorem 2.1], rewritten as a problem ]**)**.** *Suppose that we are given a d-regular graph $G = (V, E)$ for some fixed d, and suppose G has a subset of at most $\zeta n$ nodes S, for some constant $0 < \zeta < 1/2$, such that $\phi_G(S) \leq \epsilon$ for some constant $0 < \epsilon \leq 1$. Then, find as efficiently as possible a subset $S'$ of at most $\zeta n$ nodes such that $\phi_G(S) \leq \eta\epsilon$ for some "universal constant" $\eta > 0$.*

In general, computing a very good approximation of the Sse problem seems to be quite hard; the approximation ratio of the algorithm presented in (109) roughly deteriorates proportional to $\sqrt{log(1/\zeta)}$, and a $O(1)$-approximation described in (82) works only if the graph excludes two specific minors. The authors in (80) showed how to design a sub-exponential time (i.e., $O(2^{cn})$ time for some constant $c < 1$) algorithm for the above problem. As they remark, expander like graphs are somewhat easier instances of Sse for their algorithm, and it takes some non-trivial technical effort to handle the "non-expander" graphs. Note that the class of ?-hyperbolic graphs for $\delta = o(logn)$ is a non-trivial proper subclass of non-expander graphs. We show that Sse (as defined in Definition 13) can be solved in polynomial time for such a proper subclass of non-expanders.

## 5.2 polynomial time solution of Sse for $\delta$-hyperbolic graphs

**Lemma 14** (polynomial time solution of Sse for $\delta$-hyperbolic graphs when $\delta$ is sub-logarithmic and $d$ is sub-linear)**.** *Suppose that $G$ is a $d$-regular $\delta$-hyperbolic graph. Then the Sse problem for $G$ can be solved in polynomial time provided $d$ and $\delta$ satisfy:*

$$d \leq 2^{\log^{(1/3)-\rho}} n \text{ and } \delta \leq \log^{\rho} n \text{ for some constant } 0 < \rho < 1/3$$

**Remark 6.** *Computing the minimum node expansion ratio of a graph is in general NP-hard and is in fact Sse-hard to approximate within a ratio of $C\sqrt{\mathfrak{h}_G \log d}$ for some constant $C > 0$ (102). Since we show that Sse is polynomial-time solvable for $\delta$-hyperbolic graphs for some parameter* *ranges, the hardness result of (102) does not directly apply for graph classes that belong to these cases, and thus additional arguments may be needed to establish similar hardness results for these classes of graphs.*

**P**roof. Our proof is quite similar to that used for Theorem 6. But, instead of looking for smallest possible non-expansion bounds, we now relax the search and allow us to consider subsets of nodes whose expansion is just enough to satisfy the requirement. This relaxation helps us to ensure the size requirement of the subset we need to find.

We will use the construction in the proof of Theorem 1 in this proof, so *we urge the readers to familiarize themselves with the details of that proof before reading the current proof.* Note that $h_G(S) \leq \epsilon$ implies $\phi_G(S) \leq d\ h_G(S)/d \leq \epsilon$. We select the nodes $p$ and $q$ such that $\Delta = dist_G(p,q) = \log_d n = \log n/\log d$, and set $\mu = 1/2$. Note that $(360 \log n)/\Delta 2^{\Delta}\mu/(28\delta \log(2d)) < (1/\Delta)^{1-\mu}$ since

$$
\begin{aligned}
&(360 \log n)/\left(\Delta 2^{\Delta^{\mu}/(28\delta\log(2d)}\right) < (1/\Delta)^{1-\mu} \\
&\qquad \Leftarrow (360 \log d)/\left(2^{(\log n)^{1/2}/(56\delta(\log d)3/2})\right) < (\log d/\log n)^{1/2} \\
&\qquad\qquad \Leftarrow 9 + \log\log n/2 < \left(\log n)^{1/2}\right)/\left(56 \log^{(1-\rho)/2} n\right) - \log^{(1-\rho)/2} n
\end{aligned}
$$

and the last inequality clearly holds for sufficiently large $n$.

First, suppose that there exists $0 \le r \le \frac{\Delta}{3} - \alpha\Delta$ such that $h_G(B_{GC}(p,r)) = h_G(B_G(p,r)) \le \epsilon$. We return $S' = B_G(p,r)3$ as our solution, To verify the size requirement, note that

$$|B_G(p,r)| \le |B_G(p,(\Delta/3) - \alpha\Delta)| \le |B_G(p,\Delta/3)|$$
$$\sum_{i=0}^{\Delta/3} d^i < d^{(\Delta/3)+1} = dn^{1/3} < \zeta n \quad (5.1)$$

where the last inequality follows since $d \le 2\log(1/3)\rho$ n and $\zeta$ is a constant. Otherwise, no such r exists, and this implies

$$|B_G(p,(\Delta/3) - \alpha\Delta)| \ \ge \ (1+\epsilon)^{(\Delta/3)-\alpha\Delta} \ > \ (1+\epsilon)^{\Delta/4} \ \ge \ e^{\epsilon\Delta/8} \ = \ e^{\epsilon\log dn/8 = n^{\epsilon\log de/8}}$$

Now there are two major cases as follows.

**Case1:t here exists at least one path between p and q in $G_{-C}$.**

We know that $dist_{G-c}(p,q) \ge (\Delta/60)2^{\alpha\Delta/\delta}$ and (by choice of $p$) $|B_{G-c}(p, dist_{G_{-c}}(p,q)/2)| < n/2$. Let $p = u_0, u_1, ..., u_t = q$ be the nodes in successive order on a shortest path from $p$ to $q$ of length $t = dist_{G_{-C}}(p,q)$. Perform a BFS starting from p in $G_{-C}$ and let $\mathcal{L}_i$ be the sets of nodes at the $i_{th}$ level ($i.e., \forall u \in \mathcal{L}_i : dist_{G_{-C}}(p,q) = i$). Note that $|\bigcup_{j=0}^{t/2} \mathcal{L}_j| \le n/2$. Consider the levels $\mathcal{L}_0, \mathcal{L}_1, ...\mathcal{L}_{t/2}$, and partition the ordered sequence of integers $0, 1, 2, ..., t/2$ into consecutive blocks $\Delta_0.\Delta_1, ..., \Delta_{(1+(t/2))/\kappa-1}$ each of length $\eta = (8/\epsilon)\ln n$, $i.e.,$

$$\underbrace{0,1,...,\eta-1}_{\Delta_0}, \underbrace{\eta,\eta+1,...,2\eta-1}_{\Delta_1}, ........, \underbrace{(t/2)-\eta+1,(t/2)-\eta+2,...,(t/2)}_{\Delta_{(t+(t/2))/\eta-1}}$$

We claim that for every $\Delta_i$, there exists an index $i^*$ within $i^*$ (i.e., there exists an index $i\eta \le i^* \le i\eta + \eta - 1$)such that $h_G(L_{i^*}) \le \epsilon$. Suppose for the sake of contradiction that this is not true. Then, it follows that

$$\forall i\eta \le j \le i\eta + \eta - 1 : h_{G-C}(\mathcal{L}_j) \ge h_G(\mathcal{L}_j)/2 > \epsilon/2$$
$$\Rightarrow |\mathcal{L}_{i\eta+\eta-1}| > |\mathcal{L}_{i\eta}|(1 + (\epsilon/2))^\eta \ge (1 + (\epsilon/2))^{(8/\epsilon)\ln n}$$
$$\ge e^{(\epsilon/4)((8/\epsilon)\ln n)} = n^2 > n$$

which contradicts the fact that $|\bigcup_{j=0}^{t/2} \mathcal{L}_j| \le n/2$. Since $\sum_{i=0}^{(1+(t/2))/\kappa-1} |\mathcal{L}_{i^*}| < n/2$, there exists a set $\mathcal{L}_{k^*}$ such that $h_G(\mathcal{L}_{k^*}) \le \epsilon$ and

$$|\mathcal{L}_{k^i}| < \frac{n/2}{(1 + (t/2))/\kappa} < n\kappa/t < (8n \ln)/\big(\epsilon(\Delta/60)2^{\Delta^{1/2}/(7\delta \log(2d))}\big)$$
$$\le \left(48n \log^{(1/3)-\rho} n\right)/\left(\epsilon 2^{(\log^{\rho/2} n)/14}\right) < \zeta n$$

**Case2: there is no path between $p$ and $q$ in $G_{-c}$.**

In this case,were turn $B_{G_{-C}}(p, (\Delta/3) - \alpha\Delta) = B_G(p, (\Delta/3) - \alpha\Delta)$ as our solution. The size requirement follows since $|B_G(p, (\Delta/3) - \alpha\Delta)| < \zeta n$ was shown in (5.1). Note that nodes in $B_G(p, (\Delta/3) - \alpha\Delta)$ can only be connected to nodes in C, and thus

$$
\begin{aligned}
h_G(B_G(p, (\Delta/3) - \alpha\Delta)) &\leq |C|/|B_G(p, (\Delta/3) - \alpha\Delta)| \\
&\leq ((\Delta/3)d^{\alpha\Delta})/\left(n^{\epsilon \log_d e/8}\right) < n^{\alpha - (\epsilon \log_d e/8)} \log n \\
&< n^{1/(7\Delta^{1/2} \log(2d)) - (\epsilon/(8 \ln d))} \log n < \epsilon
\end{aligned}
$$

where the penultimate inequality follows since $\Delta = \omega(1)$.

In all cases, the desired subset of nodes can be found in $O(n^2 \log n)$ time. ❑

# CHAPTER 6

# GEOMETRIC CURVATURE

In this section, we describe generic geometric curvatures of graphs by using correspondence with topological objects in higher dimension.

### 6.0.1 Basic Definitions and Notations

For an undirected unweighted graph $G = (V, E)$ of $n$ nodes $v_1, \ldots, v_n$, the following notations related to $G$ are used throughout:

- $v_{i_1} \leftrightarrow v_{i_2} \leftrightarrow v_{i_3} \leftrightarrow \cdots \leftrightarrow v_{i_{k-1}} \leftrightarrow v_{i_k}$ denotes a path of *length* $k-1$ consisting of the edges $\{v_{i_1}, v_{i_2}\}$, $\{v_{i_2}, v_{i_3}\}$, $\ldots$, $\{v_{i_{k-1}}, v_{i_k}\}$.
- $\overline{u,v}$ and $\mathrm{dist}_G(u,v)$ denote a *shortest path* and the distance (*i.e.*, number of edges in $\overline{u,v}$) between nodes $u$ and $v$, respectively.
- $\mathsf{diam}(G) = \max_{v_i,v_j}\{\mathrm{dist}_G(v_i,v_j)\}$ denotes the *diameter* of $G$.
- $G \setminus E'$ denotes the graph obtained from $G$ by removing the edges in $E'$ from $E$.

A $\varepsilon$-approximate solution (or simply an $\varepsilon$-approximation) of a minimization (*resp.*, maximization) problem is a solution with an objective value no larger than (*resp.*, no smaller than) $\varepsilon$ times (*resp.*, $1/\varepsilon$ times) the value of the optimum; an algorithm of *performance* or *approximation ratio* $\varepsilon$ produces an $\varepsilon$-approximate solution. A problem is $\varepsilon$-*inapproximable* under a certain complexity-theoretic assumption means that the problem does not admit a polynomial-time $\varepsilon$-approximation algorithm assuming that the complexity-theoretic assumption is true. We will also use other *standard* definitions from structural complexity theory as readily available in any graduate level textbook on algorithms such as (70).

### 6.1 Remarks on basic topological concepts

We first review some basic concepts from topology; see introductory textbooks such as (41; 34) for further information. Although not necessary, the reader may find it useful to think of the underlying metric space as the $r$-dimensional real space $\mathbb{R}^r$ be for some integer $r > 1$.

- A subset $S \subseteq \mathbb{R}^r$ is *convex* if and only if for any $x, y \in S$, the *convex combination* of $x$ and $y$ is also in $S$.
- A set of $k+1$ points $x_0, \ldots, x_k \in \mathbb{R}^r$ are called *affinely independent* if and only if for all $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ $\sum_{j=0}^{k} \alpha_j x_j = 0$ and $\sum_{j=0}^{k} \alpha_j = 0$ implies $\alpha_0 = \cdots = \alpha_k = 0$.
- The *k-simplex* generated by a set of $k+1$ affinely independent points $x_0, \ldots, x_k \in \mathbb{R}^r$ is the subset $\mathcal{S}\big(x_0, \ldots, x_k\big)$ of $\mathbb{R}^r$ generated by *all* convex combinations of $x_0, \ldots, x_k$.
  - Each $(\ell{+}1)$-subset $\big\{x_{i_0}, \ldots, x_{i_\ell}\big\} \subseteq \big\{x_0, \ldots, x_k\big\}$ defines the $\ell$-simplex $\mathcal{S}\big(x_{i_0}, \ldots, x_{i_\ell}\big)$ that is called a *face* of dimension $\ell$ (or a *$\ell$-face*) of $\mathcal{S}\big(x_0, \ldots, x_k\big)$. A $(k-1)$-face, 1-face and 0-face is called a *facet*, an *edge* and a *node*, respectively.
- A (closed) *halfspace* is a set of points satisfying $\sum_{j=1}^{r} a_j x_j \leq b$ for some $a_1, \ldots, a_r, b \in \mathbb{R}$. The convex set obtained by a bounded non-empty intersection of a finite number of halfspaces is called a *convex polytope* (*convex polygon* in two dimensions).

- ▷ If the intersection of a halfspace and a convex polytope is a subset of the halfspace then it is called a *face* of the polytope. Of particular interests are faces of dimensions $r-1$, 1 and 0, which are called *facets*, *edges* and *nodes* of the polytope, respectively.

- ▶ A *simplicial complex* (or just a complex) is a topological space constructed by the union of simplexes via topological associations.

#### 6.1.0.1 Geometric curvature definitions

Informally, a *complex* is "glued" from nodes, edges and polygons via topological identification. We first define $k$-complex-based *Forman's combinatorial Ricci curvature* for elementary components (such as nodes, edges, triangles and higher-order cliques) as described in (32; 74; 73), and then obtain a scalar curvature that takes an appropriate linear combination of these values (via Gau$\beta$-Bonnet type theorems (12)) that correspond to the so-called *Euler characteristic* of the complex that is topologically associated with the given graph. In this regard, we consider such Euler characteristics of a graph to define geometric curvature.

To begin the topological association, we (topologically) associate a $q$-simplex with a $(q+1)$-clique $\mathcal{K}_{q+1}$; for example, 0-simplexes, 1-simplexes, 2-simplexes and 3-simplexes are associated with nodes, edges, 3-cycles (triangles) and 4-cliques, respectively. Next, we would also need the concept of an "order" of a simplex for more non-trivial topological association. Consider a $p$-face $f^p$ of a $q$-simplex. An order $d$ association of such a face, which we will denote by the notation $f_d^p$ with the additional subscript $d$, is associated with a sub-graph of *at most $d$*

nodes that is obtained by starting with $\mathcal{K}_{p+1}$ and then *optionally* replacing each edge by a path between the two nodes. For example,

- $f_d^0$ is a node of $G$ for all $d \geq 1$.
- $f_2^1$ is an edge, and $f_d^1$ for $d > 2$ is a path having at most $d$ nodes between two nodes adjacent in $G$.
- $f_3^2$ is a triangle (cycle of 3 nodes or a 3-cycle), and $f_d^2$ for $d > 3$ is obtained from 3 nodes by connecting every pair of nodes by a path such that the total number of nodes in the sub-graph is at most $d$.

Naturally, the higher the values of $p$ and $q$ are, the more complex are the topological associations. Let $\mathcal{F}_d^k$ be the set of all $f_d^k$'s in $G$ that are topologically associated. With such associations via $p$-faces of order $d$, the Euler characteristics of the graph $G = (V, E)$ and consequently the curvature can defined as

$$\mathfrak{C}_d^p(G) \stackrel{\text{def}}{=} \sum_{k=0}^{p} (-1)^k \left| \mathcal{F}_d^k \right| \tag{6.1}$$

It is easy to see that both $\mathfrak{C}_d^0(G)$ and $\mathfrak{C}_d^1(G)$ are too simplistic to be of use in practice. Thus, we consider the next higher value of $p$, namely when $p = 2$. Letting $\mathcal{C}(G)$ denote the number of cycles of at most $d+1$ nodes in $G$, we get the measure

$$\mathfrak{C}_d^2(G) = |V| - |E| + |\mathcal{C}(G)|$$

#### 6.1.0.2 Are geometric curvatures a suitable measure for real-world networks ?

The usefulness of geometric curvatures for real-world networks was demonstrated in publications such as (74; 73; 66).

Here is an example when the curvature is related to other network measures:

**Relating $\mathfrak{C}_3^2$ to other known network measures** The *global clustering coefficient* of a graph $G$ is defined as (72):

$$C \stackrel{\text{def}}{=} C(G) = \frac{3\,\Delta}{\tau} \quad \text{where} \quad \begin{array}{rcl} \Delta & = & \#\text{ of 3-cycles in } G \\ \tau & = & \#\text{ of (node-induced) sub-graphs of } G \text{ of 3 nodes and 2 edges} \end{array}$$

Letting $\mathsf{deg}_j$ denote the degree of node $v_j$, we therefore get

$$C = \frac{3\,\Delta}{\sum_{j=1}^{n} \frac{\mathsf{deg}_j(\mathsf{deg}_j - 1)}{2}} = \frac{3\,\left(\mathfrak{C}_3^2 - |V| + |E|\right)}{\left(\frac{1}{2}\sum_{j=1}^{n} \mathsf{deg}_j^2\right) - |E|}$$

For the special case when $G$ is a $d$-regular graph for some $d \geq 3$, we get

$$C = \frac{3\left(\mathfrak{C}_3^2 - |V| + \frac{d\,|V|}{2}\right)}{\frac{d^2\,|V|}{2} - \frac{d\,|V|}{2}} = \frac{6\,\mathfrak{C}_3^2}{d\,(d-1)\,|V|} + \frac{1}{d} - \frac{1}{d^2 - d}$$

Relationships between $\mathfrak{C}_3^2$ and the average clustering coefficient can also be derived in a similar manner.

# CHAPTER 7

# EFFECT OF GEOMETRIC CURVATURE ON ANOMALY DETECTION

## 7.1 Introduction

Anomaly detection problems (also called change-point detection problems) have been studied in data mining, statistics and computer science over the last several decades (mostly in non-network context) in applications such as medical condition monitoring, weather change detection and speech recognition. In recent days, however, anomaly detection problems have become increasing more relevant in the context of network science since useful insights for many complex systems in biology, finance and social science are often obtained by representing them as some type of networks. Notions of local and non-local curvatures of higher-dimensional geometric shapes and topological spaces play a fundamental role in physics and mathematics in characterizing anomalous behaviors of these higher dimensional entities. However, using curvature measures to detect anomalies in networks is not yet very common due to several reasons such as lack of preferred geometric interpretation of networks and lack of experimental evidences that may lead to specific desired curvature properties (unlike their counter-parts in theoretical physics).

## 7.2 Formalizations of two anomaly detection problems on networks

In this section, we formalize two versions of the anomaly detection problem on networks. An underlying assumption on the behind these formulations is that the graph adds/deletes *edges* only while keeping the same set of nodes.

### 7.2.1 Extremal anomaly detection for static networks

The problems in this subsection are motivated by a desire to quantify the extremal sensitivity of static networks. The basic decision question is: "*is there a subset among a set of prescribed edges whose deletion may change the network curvature significantly*?". This directly leads us to the following *decision* problem:

---

**Problem name**: Extremal Anomaly Detection Problem ($\text{Eadp}_{\mathfrak{C}}(G, \widetilde{E}, \gamma)$)

**Input**:
- A curvature measure $\mathfrak{C} : G \mapsto \mathbb{R}$
- A connected graph $G = (V, E)$, an edge subset $\widetilde{E} \subseteq E$ such that $G \setminus \widetilde{E}$ is connected and a real number $\gamma < \mathfrak{C}(G)$ (*resp.*, $\gamma > \mathfrak{C}(G)$)

**Decision question**: is there an edge subset $\widehat{E} \subseteq \widetilde{E}$ such that $\mathfrak{C}(G \setminus \widehat{E}) \leq \gamma$ (*resp.*, $\mathfrak{C}(G \setminus \widehat{E}) \geq \gamma$) ?

**Optimization question**: if the answer to the decision question is "yes" then minimize $|\widehat{E}|$

**Notation**: if the answer to the decision question is "yes" then the minimum possible value of $|\widehat{E}|$ is denoted by $\mathsf{OPT}_{\text{Eadp}_{\mathfrak{C}}}(G, \widetilde{E}, \gamma)$

---

The following comments regarding the above formulation should be noted:

▷ For the case $\gamma < \mathfrak{C}(G)$ (*resp.*, $\gamma > \mathfrak{C}(G)$) we allow $\mathfrak{C}(G \setminus \widetilde{E}) > \gamma$ (*resp.*, $\mathfrak{C}(G \setminus \widetilde{E}) < \gamma$), thus $\widehat{E} = \widetilde{E}$ need *not* be a feasible solution at all.

▷ The curvature function is only defined for connected graphs, thus we require $G \setminus \widetilde{E}$ to be connected.

▷ The edges in $E \setminus \widetilde{E}$ can be thought of as "critical" edges needed for the functionality of the network. For example, in the context of inference of minimal biological networks from indirect experimental evidences (4; 3), the set of critical edges represent direct biochemical interactions with concrete evidence.

### 7.2.2 Targeted anomaly detection for dynamic networks

These problems are primarily motivated by change-point detections between two successive discrete time steps in dynamic networks (7; 49), but they can also be applied to static networks when a subset of the final desired network *is* known. Fig. Figure 9 illustrates targeted anomaly detection for a dynamic biological network.

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**Problem name**: Targeted Anomaly Detection Problem ($\mathrm{Tadp}_{\mathfrak{C}}(G_1, G_2)$)

**Input**: • Two connected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_2 \subset E_1$

• A curvature measure $\mathfrak{C} : G \mapsto \mathbb{R}$

**Valid solution**: a subset of edges $E_3 \subseteq E_1 \setminus E_2$ such that $\mathfrak{C}(G_1 \setminus E_3) = \mathfrak{C}(G_2)$.

**Objective**: *minimize* $|E_3|$.

**Notation**: the minimum value of $|E_3|$ is denoted by $\mathsf{OPT}_{\mathrm{Tadp}_{\mathfrak{C}}}(G_1, G_2)$

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#### 7.2.3 Two examples in which curvature measures detect anomaly where other simpler measures do not

It is obviously practically impossible to compare our curvatures measures for anomaly detection with respect to *every possible* other network measure that has been used in prior research works. However, we do still provide two illustrative examples of comparing our curvature measures to the well-known *densest subgraph measure* which is defined as follows. Given a graph $G = (V, E)$, the densest subgraph measure find a subgraph $(S, E_S)$ induced by a subset of nodes $\emptyset \subset S \subseteq V$ that maximizes the ratio (density) $\rho(S) \stackrel{\text{def}}{=} \frac{|E_S|}{|S|}$. Let $\rho(G) \stackrel{\text{def}}{=} \max_{\emptyset \subset S \subseteq V} \{\rho(S)\}$ denote the density of a densest subgraph of $G$. An efficient polynomial time algorithm to compute $\rho(G)$ using a max-flow technique was first provided by Goldberg (38). *We urge the readers to review the definitions of the relevant curvature measures (in Section 1.2) and the anomaly detection problems (in Section 7.2) in case of any confusion regarding the examples we provide.*

##### 7.2.3.1 Extremal anomaly detection for a static network

Consider the extremal anomaly detection problem (Problem Eadp in Section **??**) for a network $G = (V, E)$ of 10 nodes and 20 edges as shown in Fig. Figure 8 using the geometric curvature $\mathfrak{C}_3^2$ as defined by Equation (Equation 6.1). It can be easily verified that $\mathfrak{C}_3^2(G) = 6$ and $\rho(G) = 9/4$. Let $\widetilde{E} = E$ and suppose that we set our targeted decrease of the curvature or density value to be 75% of the original value, *i.e.*, we set $\gamma = 3/4 \times \mathfrak{C}_3^2(G) = 9/2$ for the geometric curvature measure and $\gamma = 3/4 \times \rho(G) = 27/16$ for the densest subgraph measure. It is easily verified that $\mathfrak{C}_3^2(G \setminus \{e_1\}) = 0$, thus showing $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_3^2}}(G, \widetilde{E}, \gamma) = 1$. However, many more than just one edge will need to be deleted from $G$ to bring down the value of $\rho(G)$ to $27/16$.

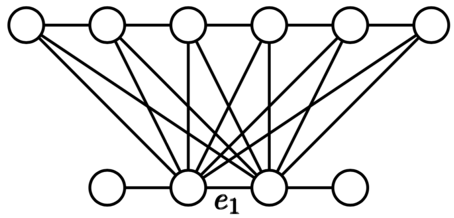


Figure 8: Toy example of extremal anomaly detection discussed in Section.

#### 7.2.3.2 Targeted anomaly detection for a dynamic biological network

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$$

$$\forall\, t = 0, 1, 2, \ldots\ldots$$

$$x_2(t+1) = x_2(t) + 0.8x_1(t)$$

$$x_3(t+1) = x_3(t) + 0.8x_2(t)$$

$$x_4(t+1) = x_4(t) + 0.8x_3(t) + \frac{0.4\,x_1(t)}{1+\mathrm{e}^{-3.66\,t+11}}$$

| $t$ | ... | 2 | 3 | ... |
|---|---|---|---|---|
| $x_1(t)$ | ... | 1.00 | 1.00 | ... |
| $x_2(t)$ | ... | 0.80 | 0.80 | ... |
| $x_3(t)$ | ... | 0.64 | 0.64 | ... |
| $x_4(t)$ | ... | 0.52 | 0.90 | ... |

**microarray output**
$> 0.60 \Rightarrow$ **ON (1)**
$< 0.55 \Rightarrow$ **OFF (0)**

$\delta =$ **value of Gromov curvature**

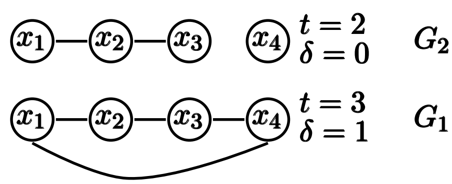


Figure 9: Toy example of targeted anomaly detection discussed in Section.

Consider the targeted anomaly detection problem (Problem Tadp in Section 7.2.2) for a dynamic biological network of 4 variables $x_1, x_2, x_3, x_4$ as shown in Fig. Figure 9, where $x_1$ affects $x_4$ with a delay, using the Gromov-hyperbolic curvature (Definition 1). Suppose that the network inference from microarray data is done by incorporating a time delay of two in the hitting-set approach of Krupa (44). It can be easily verified that $\mathfrak{C}_{\text{Gromov}}(G_1) = \rho(G_1) = 1$, $\mathfrak{C}_{\text{Gromov}}(G_2) = 0$, and $\rho(G_2) = 1/2$. Since $\mathfrak{C}_{\text{Gromov}}(G_1 \setminus \{x_1, x_4\}) = 0$ it follows that

$\mathsf{OPT}_{\mathrm{Tadp}_{\mathfrak{C}_{\mathrm{Gromov}}}}(G_1, G_2) = 1$; however, 2 edges will need to be deleted from $G_1$ to bring down the value of $\rho(G_1)$ to $\rho(G_2)$.

#### 7.2.4 Algebraic approaches for anomaly detection

In contrast to the combinatorial/geometric graph-property based approach investigated in this paper and elsewhere, an alternate approach for anomaly detection is the *algebraic tensor-decomposition based approach* studied in the contexts of dynamic social networks (67) and pathway reconstructions in cellular systems and microarray data integration from several sources (6; 59). This approach is quite different from the ones studied in this paper with its own pros and cons.

### 7.3 Computational complexity of extremal anomaly detection problems

#### 7.3.1 Why only the edge-deletion model?

In this research we add or delete edges from a network while keeping the node set the same. This scenario captures a wide variety of applications such as inducing desired outcomes in disease-related biological networks via gene knockout (65; 78), inference of minimal biological networks from indirect experimental evidences or gene perturbation data (4; 3; 71), and finding influential nodes in social and biological networks (5), to name a few. However, the node addition/deletion model or a mixture of node/edge addition/deletion model is also significant in many other applications. Although some of our complexity results can be easily extended for bounded-degree graphs to the node deletion model, we do not outline these generalizations here but leave it as a separate future research topic.

### 7.3.2 Geometric curvatures: exact and approximation algorithms for $EADP_{\mathfrak{C}_d^2}$

**Theorem 15.**

**(*a*)** *The following statements hold for* $\mathrm{E}adp_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ *when* $\gamma > \mathfrak{C}_d^2(G)$*:*

**(*a*1)** *We can decide in polynomial time the answer to the decision question (i.e., if there exists any feasible solution* $\widehat{E}$ *or not).*

**(*a*2)** *If a feasible solution exists then the following results hold:*

**(*a*2-1)** *Computing* $\mathsf{OPT}_{\mathrm{E}adp_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$ *is* NP*-hard for all* $d$ *that are multiple of* 3*.*

**(*a*2-2)** *If* $\gamma$ *is sufficient larger than* $\mathfrak{C}_d^2(G)$ *then we can design an approximation algorithm that approximates both the cardinality of the minimal set of edges for deletion and the absolute difference between the two curvature values. More precisely, if* $\gamma \geq \mathfrak{C}_d^2(G) + \left(\frac{1}{2} + \varepsilon\right)(2|\widetilde{E}| - |E|)$ *for some* $\varepsilon > 0$*, then we can find in polynomial time a subset of edges* $E_1 \subseteq \widetilde{E}$ *such that*

$$|E_1| \leq 2\,\mathsf{OPT}_{\mathrm{E}adp_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma) \quad \text{and} \quad \frac{\mathfrak{C}_d^2(G \setminus E_1) - \mathfrak{C}_d^2(G)}{\gamma - \mathfrak{C}_d^2(G)} \geq \frac{4\varepsilon}{1 + 2\varepsilon}$$

**(*b*)** *The following statements hold for* $\mathrm{E}adp_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ *when* $\gamma < \mathfrak{C}_d^2(G)$*:*

**(b*1*)** *We can decide in polynomial time the answer to the decision question (i.e., if there exists any feasible solution* $\widehat{E}$ *or not).*

**(b*2*)** *If a feasible solution exists and* $\gamma$ *is not too far below* $\mathfrak{C}_d^2(G)$ *then we can design an approximation algorithm that approximates both the cardinality of the minimal set of edges* *for deletion and the absolute difference between the two curvature values. More precisely, letting $\Delta$ denote the number of cycles of $G$ of at most $d+1$ nodes that contain at least one edge from $\widetilde{E}$, if $\gamma \geq \mathfrak{C}_d^2(G) - \frac{\Delta}{1+\varepsilon}$ for some $\varepsilon > 0$ then we can find in polynomial time a subset of edges $E_1 \subseteq \widetilde{E}$ such that*

$$|E_1| \leq 2\,\mathsf{OPT}_{\mathrm{E}adp_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma) \quad \textit{and} \quad \frac{\mathfrak{C}_d^2(G \setminus E_1) - \mathfrak{C}_d^2(G)}{\gamma - \mathfrak{C}_d^2(G)} \leq 1 - \varepsilon$$

***(*b*3)*** *If $\gamma < \mathfrak{C}_d^2(G)$ then, even if $\gamma = \mathfrak{C}_d^2(G \setminus \widetilde{E})$ (i.e., a trivial feasible solution exists), computing $\mathsf{OPT}_{\mathrm{E}adp_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$ is at least as hard as computing $\mathrm{T}adp_{\mathfrak{C}_d^2}(G_1, G_2)$ and therefore **all** the hardness results for $\mathrm{T}adp_{\mathfrak{C}_d^2}(G_1, G_2)$ apply to $\mathsf{OPT}_{\mathrm{E}adp_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$.*

### 7.3.3 Proof techniques and relevant comments regarding Theorem 15

- **(on proofs of (*a*1) and (*b*1))** After eliminating a few "easy-to-solve" sub-cases, we prove the remaining cases of **(*a*1)** and **(*b*1)** by reducing the feasibility questions to suitable minimum-cut problems; the reductions and proofs are somewhat different due to the nature of the objective function. It would of course be of interest if a single algorithm and proof can be found that covers both instances and, more importantly, if a direct and more efficient greedy algorithm can be found that *avoids* the maximum flow computation.

- **(on proofs of (*a*2-2) and (*b*2))** Our general approach to prove **(*a*2-2)** and **(*b*2)** is to formulate these problems as a series of (provably NP-hard and polynomially many) "constrained" minimum-cut problems. We start out with *two different* (but well-known) polytopes for the minimum cut problem (polytopes (Equation 7.3) and (Equation 7.3)′). Even though the polytope (Equation 7.3)$'$ is of exponential size for general graphs, it is of polynomial size for our particular minimum cut version and so we do *not* need to appeal to separation oracles for its efficient solution. We subsequently add extra constraints corresponding to a parameterized version of the minimization objective and solve the resulting augmented polytopes (polytopes (Equation 7.4) and (Equation 7.4)$'$) in polynomial time to get a fractional solution and use a simple deterministic rounding scheme to obtain the desired bounds.

- ▷ Our algorithmic approach uses a sequence of $\lceil \log_2(1 + |\widetilde{E}|) \rceil = O(\log |E|)$ linear-programming ($\mathsf{LP}$) computations by using an obvious binary search over the relevant parameter range. It would be interesting to see if we can do the same using $O(1)$ $\mathsf{LP}$ computations.
- ▷ Is the factor 2 in "$|E_1| \leq 2\,\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$" an artifact of our specific rounding scheme around the threshold of $1/2$ and perhaps can be improved using a cleverer rounding scheme? This seems *unlikely* for the case when $\gamma < \mathfrak{C}_d^2(G)$ since the inapproximability results in **(*b*3)** include a $(2 - \varepsilon)$-inapproximability assuming the unique games conjecture is true. However, this possibility cannot be ruled out for the case when $\gamma > \mathfrak{C}_d^2(G)$ since we can only prove $\mathsf{NP}$-hardness for this case.
- ▷ There are subtle but crucial differences between the rounding schemes for **(*a*2-2)** and **(*b*2)** that is essential to proving the desired bounds. To illustrate this, consider an edge $e$ with a fractional value of $1/2$ for its corresponding variable. In the rounding scheme (Equation 7.5) of **(*a*2-2)** $e$ will only *sometimes* be designated as a cut edge,

whereas in the rounding scheme (Equation 7.5)$'$ of ***(b2)*** $e$ will *always* be designated as a cut edge.

- **(on the bounds over $\boldsymbol{\gamma}$ in $(a2\text{-}2)$** If $|\widetilde{E}| \lessapprox \frac{1}{2}|E|$ then the condition on $\gamma$ *is* redundant (*i.e.*, *always* holds). Thus indeed the 2-approximation is likely to hold *unconditionally* for *practical* applications of this problem since anomaly is supposed to be caused by a *large* change in curvature by a relatively *small* number of elementary components (edges in our cases).

  Furthermore, if $|E| \leq 2|V|$ then the condition on $\gamma$ always holds *irrespective* of the value of $|\widetilde{E}|$, and the *smaller* is $|\widetilde{E}|$ with respect to $|E|$ the *better* is our approximation of the curvature difference. As a general illustration, when $\varepsilon = 1/5$ the assumptions are $\gamma \geq \mathfrak{C}_d^2(G) + \frac{7}{10}(2|\widetilde{E}| - |E|)$, and the corresponding bounds are $|E_1| \leq 2\,\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$ and $\frac{\mathfrak{C}_d^2(G \setminus E_1) - \mathfrak{C}_d^2(G)}{\gamma - \mathfrak{C}_d^2(G)} \geq \frac{4}{7}$.

- **(on the hardness proof in (*a*2-1))** Our reduction is from the *densest-k-subgraph* (D$k$S$_3$) problem. We use the reduction from the CLIQUE problem to D$k$S$_3$ detailed by Feige and Seltser in (31) which shows that D$k$S$_3$ is $\mathsf{NP}$-hard even if the degree of every node is at most 3. For convenience in doing calculations, we use the reduction of Feige and Seltser starting from the still $\mathsf{NP}$-hard version of the CLIQUE problem where the input instances are $(n-4)$-regular $n$-node graphs. Pictorially, the reduction is illustrated in Fig. Figure 10. Note that D$k$S$_3$ is *not* known to be $(1+\varepsilon)$-inapproximable assuming P$\neq$ $\mathsf{NP}$ (though it is likely to be), and thus our particular reduction cannot be generalized to $(1+\varepsilon)$-inapproximability assuming P$\neq$ $\mathsf{NP}$.

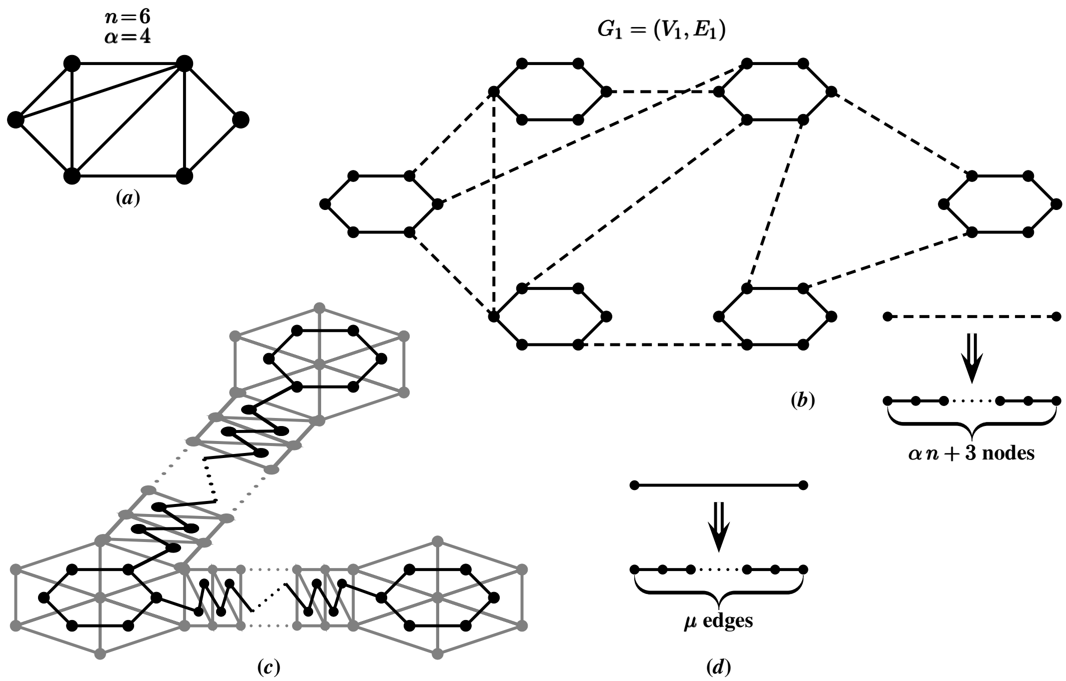


Figure 10: Illustration of the reduction in the proof of Theorem.***(a)***The original instance of the $\alpha$-CLIQUE problem on an $n$-node graph. ***(b)***Illustration of the NP-hardness reduction from $\alpha$-CLIQUE to D$k$S$_3$ by Feige and Seltser ***(c)*** Illustration of associations of nodes with unique 3-cycles such that an edge between two nodes are adjacent correspond to sharing an unique edge of their associated 3-cycles. ***(d)*** Splitting of every edge of $G_1$ into a path of $\mu$ edges for the case when $d = 3\mu$ for some integer $\mu > 1$.

#### 7.3.3.1 Proof of Theorem 15

***(a1)* and *(a2-2)*** Let the notation $\mathcal{C}(H)$ denote the set of cycles having at most $d+1$ nodes in a graph $H$. Assume $\Delta = |\mathcal{C}(G)|$ and let $\mathcal{C}(G) = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\Delta\}$; thus $\mathfrak{C}_d^2(G) = n - m + \Delta$

where $|V| = n$ and $|E| = m$. Since $d$ is fixed, $\Delta = O(n^d)$ and all the cycles in $\mathcal{C}(G)$ can be explicitly enumerated in polynomial ($O(n^d)$) time. Let $\mathcal{C}'(G) = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{\Delta'}\} \subseteq \mathcal{C}(G)$ be the set of $\Delta' \leq \Delta$ cycles in $\mathcal{C}(G)$ that involve one of more edges from $\widetilde{E}$. We first observe that the following sub-cases are easy to solve:

- If $\gamma > n - (m - |\widetilde{E}|) + \Delta$ then we can assert that there is no feasible solution. This is true because for any $E' \subseteq \widetilde{E}$ it is true that $\mathfrak{C}_d^2(G \setminus E')$ is at most $n - (m - |\widetilde{E}|) + \Delta$.
- If $\gamma \leq n - (m - |\widetilde{E}|) + \Delta$ and $\Delta' = 0$ then there exists a *trivial* optimal feasible solution of the following form:

  select any set of $m_1$ edges from $\widetilde{E}$ where $m_1$ is the least positive integer satisfying $n - (m_1 - |\widetilde{E}|) + \Delta \geq \gamma$.

Thus, we assume that $\gamma \leq n - (m - |\widetilde{E}|) + \Delta$ and $\Delta' > 0$. Consider a subset $E_1 \subseteq \widetilde{E}$ of $m_1 = |E_1| \leq |\widetilde{E}|$ edges for deletion and suppose that removal of the edges in $E_1$ removes $\Delta_1 \leq \Delta'$ cycles from $\mathcal{C}'(G)$ (*i.e.*, $|\mathcal{C}'(G \setminus E_1)| = \Delta' - \Delta_1$). Then,

$$\mathfrak{C}_d^2(G \setminus E_1) = n - (m - m_1) + (\Delta - \Delta_1) = n - m + \Delta + (m_1 - \Delta_1) = \mathfrak{C}_d^2(G) + (m_1 - \Delta_1) \tag{7.1}$$

and consequently one can observe that

$$\begin{aligned} &\mathfrak{C}_d^2(G \setminus E_1) \geq \gamma \;\equiv\; m_1 - \Delta_1 \geq \gamma - \mathfrak{C}_d^2(G) \;\equiv\; \Delta_1 - m_1 \leq \mathfrak{C}_d^2(G) - \gamma \\ &\quad \equiv\; \Delta_1 + (m - m_1) \leq \mathfrak{C}_d^2(G) - \gamma + m \;\equiv\; \Delta_1 + (|\widetilde{E}| - m_1) \leq \mathfrak{C}_d^2(G) - \gamma + |\widetilde{E}| \stackrel{\text{def}}{=} \Gamma \end{aligned} \tag{7.2}$$

Note that $\Gamma = \mathfrak{C}_d^2(G) - \gamma + |\widetilde{E}| = n - (m - |\widetilde{E}|) + \Delta - \gamma \geq 0$ and $|\widetilde{E}| - m_1$ is the number of edges in $\widetilde{E}$ that are *not* in $E_1$ and therefore not selected for deletion. Also, note that $\Gamma$ is a quantity that depends on the problem instance only and *does not change if one or more edges are deleted*. Based on this interpretation, we construct the following instance (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a (standard directed) minimum $s$-$t$ cut problem (where $\text{cap}(u, v)$ is the capacity of a directed edge $(u, v)$):

- The nodes in $\mathcal{V}$ are as follows: a source node $s$, a sink node $t$, a node (an "edge-node") $u_e$ for every edge $e \in \widetilde{E}$ and a node (a "cycle-node") $u_{\mathcal{F}_i}$ for every cycle $\mathcal{F}_i \in \mathcal{C}'(G)$. The total number of nodes is therefore $O(|\widetilde{E}| + n^d)$, *i.e.*, polynomial in $n$.
- The directed edges in $\mathcal{E}$ and their corresponding capacities are as follows:
  - For every edge $e \in \widetilde{E}$, we have a directed edge $(s, u_e)$ (an "edge-arc") of capacity $\text{cap}(s, u_e) = 1$.
  - For every cycle $\mathcal{F}_i \in \mathcal{C}'(G)$, we have a directed edge (a "cycle-arc") $(u_{\mathcal{F}_i}, t)$ of capacity $\text{cap}(u_{\mathcal{F}_i}, t) = 1$.
  - For every cycle $\mathcal{F}_i \in \mathcal{C}'(G)$ and every edge $e \in \widetilde{E}$ such that $e$ is an edge of $\mathcal{F}_i$, we have a directed edge (an "edge-cycle-arc") $(u_e, u_{\mathcal{F}_i})$ of capacity $\text{cap}(u_e, u_{\mathcal{F}_i}) = \infty$.

For an $s$-$t$ cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ of $\mathcal{G}$ (where $s \in \mathcal{S}$ and $t \notin \mathcal{S}$), let $\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S}) = \{(x, y) \,|\, x \in \mathcal{S},\, y \notin \mathcal{S}\}$ and $\text{cap}(\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) = \sum_{(x,y) \in \text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})} \text{cap}(x, y)$ denote the edges in the cut and the capacity of the cut, respectively. It is well-known how to compute a *minimum* $s$-$t$ cut of value $\Phi \stackrel{\text{def}}{=}$

$\min_{\emptyset \subset \mathcal{S} \subset \mathcal{V},\, s \in \mathcal{S},\, t \notin \mathcal{S}} \{\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S}))\}$ in polynomial time (22). The following lemma proves part $(a1)$ of the theorem.

**Lemma 16.** *There exists any feasible solution of* $\mathrm{E}adp_{\mathfrak{C}^2_d}(G, \widetilde{E}, \gamma)$ *if and only if* $\Phi \leq \Gamma$*. Moreover, if* $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ *is a minimum s-t cut of* $\mathcal{G}$ *of value* $\Phi \leq \Gamma$ *then* $\widehat{E} = \{e \,|\, u_e \in \mathcal{S}\}$ *is a feasible solution for* $\mathrm{E}adp_{\mathfrak{C}^2_d}(G, \widetilde{E}, \gamma)$*.*

**P**roof. Suppose that there exists a feasible solution $E_1 \subseteq \widetilde{E}$ with $m_1 = |E_1|$ edges for $\mathrm{Eadp}_{\mathfrak{C}^2_d}(G, \widetilde{E}, \gamma)$, and suppose that removal of the edges in $E_1$ removes $\Delta_1$ cycles from $\mathcal{C}'(G)$. Consider the cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ where

$$\mathcal{S} = \{s\} \bigcup \{u_e \,|\, e \in E_1\} \bigcup \{u_{\mathcal{F}_i} \,|\, \ \mathcal{F}_i \text{ contains at least one edge from } E_1 \ \}$$

Note that *no* edge-cycle-arc belongs to $\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ and therefore

$$\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) = |\{(s, u_e) \,|\, e \notin E_1\}| + |\{(u_{\mathcal{F}_i}, t) \,|\, \ \mathcal{F}_i \text{ contains at least one edge from } E_1 \ \}| = (|\widetilde{E}| - m_1) + \Delta_1$$

and thus by Inequality (Equation 7.2) we can conclude that

$$\mathfrak{C}^2_d(G \setminus E_1) \geq \gamma \ \equiv \ \Delta_1 + (|\widetilde{E}| - m_1) \leq \Gamma \ \equiv \ \mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) \leq \Gamma \ \Rightarrow \Phi \leq \mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) \leq \Gamma$$

For the other direction, consider a minimum $s$-$t$ cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ of $\mathcal{G}$ of value $\Phi \leq \Gamma$. Consider the solution $E_1 = \{e \,|\, u_e \in \mathcal{S}\} \subseteq \widetilde{E}$ for $\mathrm{Eadp}_{\mathfrak{C}^2_d}(G, \widetilde{E}, \gamma)$, and suppose that removal of the edges in $E_1$ removes $\Delta_1$ cycles from $\mathcal{C}'(G)$. Since $\mathcal{G}$ admits a trivial $s$-$t$ cut $(\{s\}, \mathcal{V} \setminus \{s\})$ of capacity

$m_1 < \infty$, *no* edge-cycle-arc can be an edge of *any* minimum $s$-$t$ cut of $\mathcal{G}$, *i.e.*, $\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ contains only edge-arcs or cycle-arcs. Let $E_2 = \{\mathcal{F}_j \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}$. Consider an edge $e \in E_1$ and let $\mathcal{F}_j$ be a cycle in $\mathcal{C}'(G)$ containing $e$. Since $\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ contains no edge-cycle-arc, it does *not* contain the arc $(u_e, u_{\mathcal{F}_j})$. It thus follows that the cycle-node $u_{\mathcal{F}_j}$ must also belong to $\mathcal{S}$ and thus $|E_2| = \Delta_1$. Now note that

$$\Phi = |\{u_e \,|\, u_e \notin \mathcal{S}\}| + |\{u_{\mathcal{F}_j} \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}| = (|\widetilde{E}| - |E_1|) + \Delta_1 \leq \Gamma = \mathfrak{C}_d^2(G) - \gamma + |\widetilde{E}|$$

$$\equiv \mathfrak{C}_d^2(G \setminus E_1) = \mathfrak{C}_d^2(G) + |E_1| - \Delta_1 \geq \gamma$$

❑

This completes a proof for ($a$1). We now prove ($a$2-2). Let $\widehat{E} \subseteq \widetilde{E}$ be an optimal solution of the optimization version of $\mathrm{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ having $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$ nodes. Note that $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma) \in \{1, 2, \ldots, |\widetilde{E}|\}$ and thus in polynomial time we can "guess" every possible value of $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$, solve the corresponding optimization problem with this additional constraint, and take the best of these solutions. In other words, it suffices if we can find, under the assumption that $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma) = \kappa$ for some $\kappa \in \{1, 2, \ldots, |\widetilde{E}|\}$, find a solution $E_1 \subseteq \widetilde{E}$ satisfying the claims in ($a$2-2).

We showed in part ($a$1) that the feasibility problem can be reduced to finding a minimum $s$-$t$ cut of the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Notice that $\mathcal{G}$ is acyclic, and every path between $s$ and $t$ has *exactly three* directed edges, namely an edge-arc followed by a edge-cycle-arc followed by a cycle-arc. The minimum $s$-$t$ cut problem for a graph has a well-known associated convex polytope of *polynomial* size (*e*.g., see (70, pp. 98-99)). Letting $p_\beta$ to be the variable corresponding to each node $\beta \in \mathcal{V}$, and $d_\alpha$ to be the variable associated with the edge $\alpha \in \mathcal{E}$, this minimum *s*-*t* cut polytope for the graph $\mathcal{G}$ is as follows:

$$
\begin{array}{lll}
\text{minimize} & \sum_{\alpha\in\mathcal{E}} \operatorname{cap}(\alpha) d_\alpha = \sum_{\alpha\in\mathcal{E},\ \ \alpha \text{ is not edge-cycle-arc}} d_\alpha + \sum_{\alpha\in\mathcal{E},\ \ \alpha \text{ is edge-cycle-arc}} \infty \times d_\alpha & \\
\text{subject to} & d_\alpha \geq p_\beta - p_\xi & \text{for every edge } \alpha = (\beta, \xi) \in \mathcal{E} \\
& p_s - p_t \geq 1 & \\
& 0 \leq p_\beta \leq 1 & \text{for every node } \beta \in \mathcal{V} \\
& 0 \leq d_\alpha \leq 1 & \text{for every edge } \alpha \in \mathcal{E}
\end{array}
\tag{7.3}
$$

It is well-known that all extreme-point solutions of (Equation 7.3) are integral. An integral solution of (Equation 7.3) generates a *s*-*t* cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ by letting $\mathcal{S} = \{\beta \,|\, p_\beta = 1\}$ and $\operatorname{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S}) = \{\alpha \,|\, d_\alpha = 1\}$. For our case, we have an additional constraint in that the number of edges to be deleted from $\widetilde{E}$ is $\kappa$, which motivates us to formulate the following polytope for our problem:

$$
\begin{array}{rll}
\text{minimize} & \sum_{\alpha\in\mathcal{E}} \text{cap}(\alpha) d_\alpha = \sum_{\alpha\in\mathcal{E},\ \ \alpha \text{ is not edge-cycle-arc}} d_\alpha + \sum_{\alpha\in\mathcal{E},\ \ \alpha \text{ is edge-cycle-arc}} \infty \times d_\alpha & \\
\text{subject to} & d_\alpha \geq p_\beta - p_\xi \qquad \text{for every edge } \alpha = (\beta,\xi) \in \mathcal{E} & \\
& p_s - p_t \geq 1 & \\
& 0 \leq p_\beta \leq 1 \qquad \text{for every node } \beta \in \mathcal{V} & \\
& 0 \leq d_\alpha \leq 1 \qquad \text{for every edge } \alpha \in \mathcal{E} & \\
& \sum_{u_e\in\mathcal{V}} p_{u_e} = \kappa &
\end{array}
\tag{7.4}
$$

Let $\mathsf{OPT}_{(\text{Equation 7.4})}$ denote the optimal objective value of (Equation 7.4).

**Lemma 17.** $\mathsf{OPT}_{(\text{Equation 7.4})} \leq \Gamma$.

**P**roof. Suppose that removal of the edges in the optimal solution $\widehat{E}$ removes $\widehat{\Delta} \leq \Delta'$ cycles from $\mathcal{C}'(G)$. We construct the following solution of (Equation 7.4) with respect to the optimal solution $\widehat{E}$ of $\text{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ having $|\widehat{E}| = \kappa$ nodes:

$$
\mathcal{S} = \{s\} \bigcup \{u_e \,|\, e \in \widehat{E}\} \bigcup \left\{ u_{\mathcal{F}_i} \,|\, \ \mathcal{F}_i \text{ contains at least one edge from } \widehat{E} \ \right\}
$$

$$
p_\beta = \begin{cases} 1, & \text{if } \beta \in \mathcal{S} \\ 0, & \text{otherwise} \end{cases} \qquad d_\alpha = \begin{cases} 1, & \text{if } \alpha \in \text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S}) \\ 0, & \text{otherwise} \end{cases}
$$

It can be verified as follows that this is indeed a feasible solution of (Equation 7.4):

- Since $p_s = 1$ and $p_t = 0$, it follows that $p_s - p_t \geq 1$ is satisfied.

- No edge-cycle-arc belongs to $\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$. Thus, if $\alpha = (\beta, \xi)$ is an edge-cycle-arc then $d_\alpha = 0$ and it is not the case that $p_\beta = 1$ and $p_\xi = 0$. Thus for every edge-cycle-arc $\alpha$ the constraint $d_\alpha \geq p_\beta - p_\xi$ is satisfied.

- Consider an edge-arc $\alpha = (s, u_e)$; note that $p_s = 1$. If $u_e \in \mathcal{S}$ that $p_{u_e} = 1$ and $d_\alpha = 0$, otherwise $p_{u_e} = 0$ and $d_\alpha = 1$. In both cases, the constraint $d_\alpha \geq p_\beta - p_\xi$ is satisfied. The case of a cycle-arc is similar.

- The constraint $\sum_{u_e \in \mathcal{V}} p_{u_e} = \kappa$ is *trivially* satisfied since $|\widetilde{E}| = \kappa$ by our assumption.

Note that $\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ does *not* contain any edge-cycle-arcs. Thus, the objective value of this solution is

$$\sum_{a \in \mathcal{E}} d_a = \sum_{a \in \text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})} d_a = |\{u_e \,|\, u_e \notin \mathcal{S}\}| + |\{u_{\mathcal{F}_j} \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}| = (|\widetilde{E}| - |\widehat{E}|) + \widehat{\Delta} \leq \Gamma$$

where the last inequality follows by (Equation 7.2) since $\mathfrak{C}_d^2(G \setminus \widehat{E}) \geq \gamma$. ❏

Given a polynomial-time obtainable optimal solution values $\left\{d_\alpha^*, p_\beta^* \,|\, \alpha \in \mathcal{E},\, \beta \in \mathcal{V}\right\}$ of the variables in (Equation 7.4), consider the following simple rounding procedure, the corresponding cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ of $\mathcal{G}$, and the corresponding solution $E_1 \subseteq \widetilde{E}$ of $\text{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$:

$$\hat{p}_\beta = \begin{cases} 1, & \text{if } p_\beta^* \geq 1/2 \\ 0, & \text{otherwise} \end{cases} \qquad \mathcal{S} = \{\beta \in \mathcal{V} \,|\, \hat{p}_\beta = 1\} \qquad E_1 = \{e \,|\, u_e \in \mathcal{S}\} \tag{7.5}$$

Note that in inequalities $p_s - p_t \geq 1$, $0 \leq p_s \leq 1$ and $0 \leq p_t \leq 1$ ensures that $p_s^* = 1$ and $p_t^* = 0$.

**Lemma 18.** $|E_1| \leq 2\,\kappa$.

**P**roof. $|E_1| = |\{u_e \,|\, p^*_{u_e} \geq 1/2\}| \leq 2 \sum_{u_e \in \mathcal{V}} p^*_{u_e} = 2\,\kappa$. ❑

**Lemma 19.** $\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) \leq 2\,\mathsf{OPT}_{(\text{Equation 7.4})} \leq 2\,\Gamma$.

**P**roof. Since $\mathrm{cap}(\alpha) = \infty$ and $\mathsf{OPT}_{(\text{Equation 7.4})} \leq \Gamma < \infty$, $d^*_\alpha = 0$ for any edge-cycle-arc $\alpha = (u_e, u_{\mathcal{F}_j})$, and thus $p^*_{u_e} \leq p^*_{u_{\mathcal{F}_j}}$ for such an edge. It therefore follows that

$$p^*_{u_e} \geq 1/2 \Rightarrow p^*_{u_{\mathcal{F}_j}} \geq 1/2 \quad \equiv \quad \hat{p}_{u_e} = 1 \Rightarrow \hat{p}_{u_{\mathcal{F}_j}} = 1$$

Thus, no edge-cycle-arc belongs to $\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$. Thus using Lemma 17 it follows that

$$\begin{aligned}
&\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) = |\{(s, u_e) \,|\, \hat{p}_{u_e} = 0\}| + |\{(u_{\mathcal{F}_j}, t) \,|\, \hat{p}_{u_{\mathcal{F}_j}} = 1\}| \\
&= |\{(s, u_e) \,|\, p^*_{u_e} < 1/2\}| + |\{(u_{\mathcal{F}_j}, t) \,|\, p^*_{u_{\mathcal{F}_j}} \geq 1/2\}| \leq 2 \sum_{p^*_{u_e} < 1/2} (p^*_s - p^*_{u_e}) + 2 \sum_{p^*_{\mathcal{F}_j} \geq 1/2} (p^*_{\mathcal{F}_j} - p^*_t) \\
&\qquad \leq 2 \sum_{p^*_{u_e} < 1/2} d^*_{s, p_{u_e}} + 2 \sum_{p^*_{\mathcal{F}_j} \geq 1/2} d^*_{s, p_{\mathcal{F}_j}} < 2 \sum_{\alpha \in \mathcal{E}} \mathrm{cap}(\alpha) d^*_\alpha \leq 2\,\Gamma
\end{aligned}$$

❑

Since no edge-cycle-arc belongs to $\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$, if an edge $e \in E_1$ is involved in a cycle $\mathcal{F}_j \in \mathcal{C}'(G)$ then it must be the case that $(u_e, u_{\mathcal{F}_j}) \notin \mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$. Thus, letting $m_1 = |E_1|$

and $\Delta_1 = |\{\mathcal{F}_j \in \mathcal{C}'(G) \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}|$, the claimed bound on $\mathfrak{C}_d^2(G \setminus E_1)$ can be shown as follows using Lemma 19:

$$\begin{aligned}
& \mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) = (m - m_1) + \Delta_1 \le 2\,\Gamma = 2\,\mathfrak{C}_d^2(G) - 2\,\gamma + 2\,|\widetilde{E}| \\
\Rightarrow\quad & \mathfrak{C}_d^2(G \setminus E_1) = \mathfrak{C}_d^2(G) + m_1 - \Delta_1 \ge 2\,\gamma - \mathfrak{C}_d^2(G) - (2\,|\widetilde{E}| - m), \quad \text{by (Equation 7.1)} \\
\Rightarrow\quad & \frac{\mathfrak{C}_d^2(G \setminus E_1) - \mathfrak{C}_d^2(G)}{\gamma - \mathfrak{C}_d^2(G)} \ge 2 - \frac{2\,|\widetilde{E}| - m}{\gamma - \mathfrak{C}_d^2(G)} \ge 2 - \frac{1}{\frac{1}{2} + \varepsilon} = \frac{4\varepsilon}{1 + 2\varepsilon}
\end{aligned}$$

**(*a*2-1)** The decision version of computing $\mathsf{OPT}_{\mathrm{Eadp}_{\mathfrak{C}_d^2}}(G, \widetilde{E}, \gamma)$ is as follows: "given an instance $\mathrm{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ and an integer $\kappa > 0$, is there a solution $\widehat{E} \subseteq \widetilde{E}$ satisfying $|\widehat{E}| \le \kappa$ ?". We first consider the case of $d = 3$. We will reduce from the decision version of the D$k$S$_3$ problem which is defined as follows: *given an undirected graph $G_1 = (V_1, E_1)$ where the degree of every node is either 2 or 3 and two integers $k$ and $t$, is there a (node-induced) subgraph of $G_1$ that has $k$ nodes and at least $t$ edges*? Assuming that their reduction is done from the clique problem on a $(n-4)$-regular $n$-node graph (which is $\mathsf{NP}$-hard (19)), the proof of Feige and Seltser in (31) shows that D$k$S$_3$ is $\mathsf{NP}$-complete for the following parameter values (for some integer $\sqrt{n} < \alpha \le n - 4$):

$$|V_1| = n^2 + (\alpha\, n + 1)\left(\frac{n\,(n-4)}{2}\right), \qquad |E_1| = |V_1| + \frac{n\,(n-4)}{2}$$
$$k = \alpha\, n + \binom{\alpha}{2}(\alpha\, n + 1), \qquad t = \alpha\, n + \binom{\alpha}{2}(\alpha\, n + 2)$$

We briefly review the reduction of Feige and Seltser in (31) as needed from our purpose. Their reduction is from the $\alpha$-CLIQUE problem which is defined as follows: "given a graph of $n$ nodes, does there exist a clique (complete subgraph) of size $\alpha$?". Given an instance of $\alpha$-CLIQUE, they create an instance $G_1 = (V_1, E_1)$ of D$k$S$_3$ (with the parameter values shown above) in which every node is replaced by a cycle of $n$ edges and an edge between two nodes is replaces by a path of length $\alpha\, n + 3$ between two unique nodes of the two cycles corresponding to the two nodes (see Fig. Figure 10 $(a)$–$(b)$ for an illustration). Given such an instance of D$k$S$_3$ with $V_1 = \{u_1, \ldots, u_{|V_1|}\}$ and $E_1 = \{a_1, \ldots, a_{|E_1|}\}$, we create an instance of $\mathrm{Eadp}_{\mathfrak{C}_3^2}(G, \widetilde{E}, \gamma)$ as follows:

- We associate each node $u_i \in V_1$ with a triangle (the "node triangle") $\mathcal{L}_i$ of 3 nodes in $V$ such that every edge $\{u_i, u_j\} \in E_1$ is mapped to a *unique* edge (the "shared edge") $e_{u_i,u_j} \in E$ that is shared by $\mathcal{L}_i$ and $\mathcal{L}_j$ (see Fig. Figure 10 $(c)$). Since in the reduction of Feige and Seltser (31) all nodes have degree 2 or 3 and two degree 3 nodes do not share more than one edge such a node-triangle association is possible. We set $\widetilde{E}$ to be the set of *all* shared edges; note that $|\widetilde{E}| = |E_1|$. Let $\mathcal{L} = \{v_1, v_2, \ldots\ldots\}$ be the set of all nodes in the that appear in any node triangle; note that $|\mathcal{L}| < 3\,|V_1|$.
- To maintain connectivity after all edges in $\widetilde{E}$ are deleted, we introduce $3|\mathcal{L}|+1$ new nodes $\{w_0\} \cup \big\{w_{i,j} \,|\, i \in \{1, 2, \ldots, |\mathcal{L}|\},\, j \in \{1, 2, 3\}\big\}$ and $4\,|\mathcal{L}|$ new edges

$$\Big\{\{w_0, w_{j,1}\},\, \{w_{j,1}, w_{j,2}\},\, \{w_{j,2}, w_{j,3}\},\, \{w_{j,3}, v_j\} \,|\, j \in \{1, 2, \ldots, |\mathcal{L}|\}\Big\}$$

- We set $\gamma = \mathfrak{C}_3^2(G) + (t-k) = \mathfrak{C}_3^2(G) + \binom{\alpha}{2}$.

First, we show that $\mathrm{Eadp}_{\mathfrak{C}_3^2}(G, \widetilde{E}, \gamma)$ indeed *has* a trivial feasible solution, namely a solution that contains all the edges from $\widetilde{E}$. The number of triangles $\Delta'$ that include one or more edges from $\widetilde{E}$ is precisely $|V_1|$ and thus using (Equation 7.1) we get:

$$\mathfrak{C}_3^2(G \setminus \widetilde{E}) = \mathfrak{C}_3^2(G) + |\widetilde{E}| - |\Delta'| = \mathfrak{C}_3^2(G) + |E_1| - |V_1| = \mathfrak{C}_3^2(G) + \frac{n(n-4)}{2} > \mathfrak{C}_3^2(G) + \binom{\alpha}{2} = \gamma$$

where the last inequality follows since $\alpha \leq n - 4$. The following lemma completes our proof.

**Lemma 20.** *$G_1$ has a subgraph of $k$ nodes and at least $t$ edges if and only if the instance of* $\mathrm{E}adp_{\mathfrak{C}_3^2}(G, \widetilde{E}, \gamma)$ *constructed above has a solution $\widehat{E} \subseteq \widetilde{E}$ satisfying $|\widehat{E}| \leq t$.*

**P**roof. Suppose that $G_1$ has $k$ nodes $u_1, u_2, \ldots, u_k$ such that the subgraph $H_1$ induced by these nodes has $t' \geq t$ edges. Remove an arbitrary set of $t' - t$ edges from $H_1$ to obtain a subgraph $H_1' = (V_1', E_1')$, and let $\widehat{E} = \{e_{u_i,u_j} \,|\, i, j \in \{1, 2, \ldots, k\}, \{u_i, u_j\} \in E_1'\}$. Obviously, $|\widehat{E}| = t$. Consider the triangle $\mathcal{L}_i$ corresponding to a node $u_i \in \{u_1, u_k, \ldots, u_k\}$, and let $I(\mathcal{L}_i)$ be the 0-1 *indicator variable* denoting if $\mathcal{L}_i$ is eliminated by removing the edges in $\widehat{E}$, *i.e.*, $I(\mathcal{L}_i) = 1$ (resp., $I(\mathcal{L}_i) = 0$) if and only if $\mathcal{L}_i$ *is* eliminated (resp., is *not* eliminated) by removing the edges in $\widehat{E}$. Note that the triangle $\mathcal{L}_i$ gets removed if and only if there exists another node $u_j \in \{u_1, u_k, \ldots, u_k\}$ such that $\{u_i, u_j\} \in E_1$. Thus, the total number of triangles eliminated by removing the edges in $\widehat{E}$ is at most $\sum_{i=1}^{k} I(\mathcal{L}_i) \leq k$ and consequently

$$\mathfrak{C}_3^2(G \setminus E') = \mathfrak{C}_3^2(G) + |\widehat{E}| - \sum_{i=1}^{k} I(\mathcal{L}_i) \geq \mathfrak{C}_3^2(G) + t - k = \gamma$$

Conversely, suppose that the instance of $\text{Eadp}_{\mathfrak{C}_3^2}(G, \widetilde{E}, \gamma)$ has a solution $\widehat{E} \subseteq \widetilde{E}$ satisfying $|\widehat{E}| = \widehat{t} \leq t$. Let $V_1' = \{u_j \,|\, \mathcal{L}_j$ is removed by removing one of more edges from $\widehat{E}\}$. Using (Equation 7.1) we get

$$\mathfrak{C}_3^2(G \setminus \widehat{E}) \geq \gamma = \mathfrak{C}_3^2(G) + t - k \Rightarrow \widehat{t} - |V_1'| \geq t - k \tag{7.6}$$

Let $H_1' = (V_1', E_1')$ be the subgraph of $G_1$ induced by the nodes in $V_1'$. Clearly, $|E_1'| \geq \widehat{t}$. If $|V_1| < k$ then we use the following procedure to add $k - |V_1'|$ nodes:

$V_1'' \leftarrow V_1'$

**while** $|V_1''| \neq k$ **do**

select a node $u_j \notin V_1''$ connected to one or more nodes in $V_1''$ and add $u_j$ to $V_1''$

Let $H_1'' = (V_1'', E_1'')$ be the subgraph of $G_1$ induced by the nodes in $V_1''$. Note that $|V_1''| = k$ and $|E_1''| \geq |E_1'| + (k - |V_1'|)$, and thus using (Equation 7.6) we get

$$|E_1''| \geq |E_1'| + (k - |V_1'|) \geq \widehat{t} + (k - |V_1'|) \geq t$$

❑

This concludes the proof for $d = 3$. For the case when $d = 3\mu$ for some integer $\mu > 1$, the same reduction can be used provide we split *every* edge of $G_1$ into a path of length $\mu$ by using new $\mu - 1$ nodes (see Fig. Figure 10 $(d)$).

***(b1)* and *(b2)* We will reuse the notations used in the proof of *(a)*.** We modify the proof and the proof technique in ***(a1)*** for the proof of ***(b1)***. We now observe that the following sub-cases are easy to solve:

- If $\gamma < n - m + 1 + \Delta - \Delta'$ then we can assert that there is no feasible solution. This is true because for any $E' \subseteq \widetilde{E}$ it is true that $\mathfrak{C}^2_d(G \setminus E')$ is at least $n - (m-1) + (\Delta - \Delta')$.
- If $\gamma \geq n - m + 1 + \Delta - \Delta'$ and $\Delta' = 0$ then there exists a *trivial* optimal feasible solution of the following form: *select any set of $m_1$ edges from $\widetilde{E}$ where $m_1$ is the largest positive integer satisfying $n - m_1 + 1 + \Delta \leq \gamma$.*

Thus, we assume that $\gamma \geq n - m + 1 + \Delta - \Delta'$ and $\Delta' > 0$. (Equation 7.1) still holds, but (Equation 7.2) is now rewritten as (note that $\Gamma > 0$):

$$\mathfrak{C}^2_d(G \setminus E_1) \leq \gamma \;\equiv\; m_1 - \Delta_1 \leq \gamma - \mathfrak{C}^2_d(G) \;\equiv\; m_1 + (\Delta' - \Delta_1) \leq \gamma - \mathfrak{C}^2_d(G) + \Delta' \stackrel{\text{def}}{=} \Gamma$$

(Equation 7.2)$'$

The nodes in the di-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are same as before, but the directed edges are modified as follows:

- For every edge $e \in \widetilde{E}$, we have an edge $(u_e, t)$ (an "edge-arc") of capacity $\text{cap}(u_e, t) = 1$.
- For every cycle $\mathcal{F}_i \in \mathcal{C}'(G)$, we have an edge (a "cycle-arc") $(s, u_{\mathcal{F}_i})$ of capacity $\text{cap}(s, u_{\mathcal{F}_i}) = 1$.
- For every cycle $\mathcal{F}_i \in \mathcal{C}'(G)$ and every edge $e \in \widetilde{E}$ such that $e$ is an edge of $\mathcal{F}_i$, we have a directed edge (an "cycle-edge-arc") $(u_{\mathcal{F}_i}, u_e)$ of capacity $\text{cap}(u_{\mathcal{F}_i}, u_e) = \infty$.

Corresponding to a feasible solution $E_1$ of $m_1$ edges for $\mathrm{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$ that removes $\Delta_1$ cycles, exactly the same cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ described before includes *no* cycle-edge-arcs and has a capacity of

$$
\begin{aligned}
\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) &= |\{(s, u_{\mathcal{F}_i}) \,|\, \mathcal{F}_i \text{ does not contain one or mores edges from } E_1 \}| + |\{(u_e, t) \,|\, e \in E_1\}| \\
&= (\Delta' - \Delta_1) + m_1
\end{aligned}
$$

Therefore $\mathfrak{C}_d^2(G \setminus E_1) \leq \gamma$ implies $\mathrm{cap}(\mathrm{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})) \leq \Gamma$, as desired. Conversely, given a minimum $s$-$t$ cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ of $\mathcal{G}$ of value $\Phi \leq \Gamma$, we consider the solution $E_1 = \{e \,|\, u_e \in \mathcal{S}\}$ for $\mathrm{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$. Let $\Psi = \{\mathcal{F}_j \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}$ and let $\Upsilon$ be the cycles from $\mathcal{C}'(G)$ that are removed by deletion of the edges in $E_1$. Since *no* cycle-edge-arc (of infinite capacity) can be an edge of the minimum $s$-$t$ cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$, $\Psi$ is a subset of $\Upsilon$. We therefore have

$$
\Phi = |\{u_{\mathcal{F}_j} \,|\, u_{\mathcal{F}_j} \notin \mathcal{S}\}| + |\{u_e \,|\, u_e \in \mathcal{S}\}| = (\Delta' - |\Psi|) + |E_1| \leq \Gamma \Rightarrow (\Delta' - |\Upsilon|) + |E_1| \leq \Gamma
$$

and the last inequality implies $\mathfrak{C}_d^2(G \setminus E_1) \leq \gamma$.

This completes a proof for $(b1)$. We now prove $(b2)$. We use an approach similar to that in $(a2)$ but with a *different polytope* for the minimum $s$-$t$ cut of $\mathcal{G}$. Let $\mathcal{P}$ be the set of all possible

$s$-$t$ paths in $\mathcal{G}$. Then, an alternate polytope for the minimum $s$-$t$ cut is as follows (cf. see (20.2) in (70, p. 168)):

$$\begin{array}{rlll} \text{minimize} & \sum_{\alpha\in\mathcal{E}} \text{cap}(\alpha) d_\alpha & & \\ \text{subject to} & \sum_{\alpha\in p} d_\alpha \geq 1 & \text{for every } s\text{-}t \text{ path } p \in \mathcal{P} & \\ & 0 \leq d_\alpha \leq 1 & \text{for every edge } \alpha \in \mathcal{E} & \end{array} \qquad \text{(Equation 7.3)}'$$

An integral solution of (Equation 7.3)$'$ generates a $s$-$t$ cut $(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ by letting $\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S}) = \{\alpha \,|\, d_\alpha = 1\}$. Since the capacity of any cycle-edge-arc in $\infty$, $\text{cut}(\mathcal{S}, \mathcal{V}\setminus\mathcal{S})$ contains only cycle-arcs or edge-arcs, and the number of edge-arcs in $\text{cut}(\mathcal{S}, \mathcal{V} \setminus \mathcal{S})$ for an integral solution is precisely the number of edge-nodes in $\mathcal{S}$. This motivates us to formulate the following polytope for our problem to ensure that integral solutions constrain the number of edges to be deleted from $\widetilde{E}$ to be $\kappa$:

$$\begin{array}{rlll} \text{minimize} & \sum_{\alpha\in\mathcal{E}} \text{cap}(\alpha) d_\alpha & & \\ \text{subject to} & \sum_{\alpha\in p} d_\alpha \geq 1 & \text{for every } s\text{-}t \text{ path } p \in \mathcal{P} & \\ & 0 \leq d_\alpha \leq 1 & \text{for every edge } \alpha \in \mathcal{E} & \\ & \sum_{e\in\widetilde{E}} d_{(u_e,t)} = \kappa & & \end{array} \qquad \text{(Equation 7.4)}'$$

For our problem, $|\mathcal{P}| < \binom{|\mathcal{V}|}{3}$ and thus (Equation 7.4)$'$ can be solved in polynomial time. Let $\mathsf{OPT}_{\text{(Equation 7.4)}'}$ denote the optimal objective value of (Equation 7.4)$'$. It is very easy to see that $\mathsf{OPT}_{\text{(Equation 7.4)}'} \leq \Gamma$: assuming that deletion of the $\kappa$ edges in the optimal solution $\widehat{E}$

removes $\widehat{\Delta}$ cycles from $\mathcal{C}'(G)$, we set $d_\alpha = \begin{cases} 1, & \alpha = (u_{\mathcal{F}_j}, u_e),\ e \in \widehat{E} \\ 0, & \text{otherwise} \end{cases}$ to construct a feasible solution of (Equation 7.4)$'$ of objective value

$$\sum_{\alpha\in\mathcal{E}} d_\alpha = |\{u_{\mathcal{F}_j} \,|\, d_{(s,u_{\mathcal{F}_j})} = 1\}| + |\{u_e \,|\, d_{(u_e,t)} = 1\}| = (\Delta' - \widehat{\Delta}) + |\widetilde{E}| \leq \Gamma$$

where the last inequality follows by (Equation 7.2)$'$ since $\mathfrak{C}_d^2(G\backslash\widehat{E}) \leq \gamma$. Note that the constraint $\sum_{e\in\widetilde{E}} d_{(u_e,t)} = \kappa$ is satisfied since $\sum_{e\in\widetilde{E}} d_{(u_e,t)} = |\{d_{(u_e,t)} \,|\, d_{(u_e,t)} = 1\}| = |\{e \,|\, e \in \widehat{E}\}| = \kappa$.

Given a polynomial-time obtainable optimal solution values $\{d^*_\alpha \,|\, \alpha \in \mathcal{E}\}$ of the variables in (Equation 7.4)$'$, consider the following simple rounding procedure, the corresponding cut $(\mathcal{S}, \mathcal{V}\setminus\mathcal{S})$ of $\mathcal{G}$, and the corresponding solution $E_1 \subseteq \widetilde{E}$ of $\text{Eadp}_{\mathfrak{C}_d^2}(G, \widetilde{E}, \gamma)$:

$$\hat{d}_\alpha = \begin{cases} 1, & \text{if } d^*_\alpha \geq 1/2 \\ 0, & \text{otherwise} \end{cases} \qquad E' = \{\alpha \,|\, \hat{d}_\alpha = 1\} \qquad E_1 = \{e \,|\, (u_e,t) \in E'\} \qquad \text{(Equation 7.5)}'$$

**Lemma 21.** *$E'$ is indeed a s-t cut of $\mathcal{G}$ and $E'$ does not contain any cycle-edge-arc.*

**P**roof. Since the capacity of any cycle-edge-arc $\alpha$ in $\infty$, $d^*_\alpha = 0$ and therefore $\alpha \notin E'$. To see that $E'$ is indeed a $s$-$t$ cut, consider any $s$-$t$ path $(s, u_{\mathcal{F}_j}), (u_{\mathcal{F}_j}, u_e), (u_e, t)$. Since $d^*_{(u_{\mathcal{F}_j},u_e)} =$, we have $d_{(s,u_{\mathcal{F}_j})} + d_{(u_{\mathcal{F}_j},u_e)} + d_{(u_e,t)} = d_{(s,u_{\mathcal{F}_j})} + d_{(u_e,t)} \geq 1$, which implies $\max\{d_{(s,u_{\mathcal{F}_j})}, d_{(u_e,t)}\} \geq 1/2$, putting at least one edge of the path in $E'$ for deletion. ❑

Note that $|E_1| = |\{e \,|\, d^*_{(u_e,t)} \geq 1/2\}| \leq 2\,\sum_{e\in\widetilde{E}} d^*_{(u_e,t)} = 2\,\kappa$, as desired. Let $(\mathcal{S},\mathcal{V}\setminus\mathcal{S})$ be the $s$-$t$ cut such that $\mathrm{cut}(\mathcal{S},\mathcal{V}\setminus\mathcal{S}) = E'$. It thus follows that

$$\mathrm{cap}(\mathrm{cut}(\mathcal{S},\mathcal{V}\setminus\mathcal{S})) = |E'| = |\{\alpha \,|\, d^*_\alpha \geq 1/2\}| \leq 2\sum_{\alpha\in\mathcal{E}} \mathrm{cap}(\alpha) d^*_\alpha = 2\,\mathsf{OPT}_{(\text{Equation 7.4})'} \leq 2\,\Gamma \tag{7.7}$$

Let $\Psi = \{\mathcal{F}_j \,|\, u_{\mathcal{F}_j} \in \mathcal{S}\}$ and let $\Upsilon$ be the cycles from $\mathcal{C}'(G)$ that are removed by deletion of the edges in $E_1$. Since *no* cycle-edge-arc (of infinite capacity) can be an edge of the minimum $s$-$t$ cut $(\mathcal{S},\mathcal{V}\setminus\mathcal{S})$, $\Psi$ is a subset of $\Upsilon$. The claimed bound on $\mathfrak{C}^2_d(G\setminus E_1)$ can now be shown as follows using (Equation 7.7):

$$\begin{aligned}
& \mathrm{cap}(\mathrm{cut}(\mathcal{S},\mathcal{V}\setminus\mathcal{S})) = |\{u_{\mathcal{F}_j} \,|\, u_{\mathcal{F}_j} \notin \mathcal{S}\}| + |\{u_e \,|\, u_e \in \mathcal{S}\}| = (\Delta' - |\Psi|) + |E_1| \leq 2\,\Gamma \\
\Rightarrow\quad & (\Delta' - |\Upsilon|) + |E_1| \leq 2\,\Gamma = 2\,\gamma - 2\,\mathfrak{C}^2_d(G) + 2\,\Delta' \\
\Rightarrow\quad & \mathfrak{C}^2_d(G\setminus E_1) - \mathfrak{C}^2_d(G) = |E_1| - |\Upsilon| \leq 2\,\gamma - 2\,\mathfrak{C}^2_d(G) + \Delta' \\
\Rightarrow\quad & \tfrac{\mathfrak{C}^2_d(G\setminus E_1) - \mathfrak{C}^2_d(G)}{\gamma - \mathfrak{C}^2_d(G)} \leq 2 + \tfrac{\Delta'}{\gamma - \mathfrak{C}^2_d(G)} \leq 1 - \varepsilon
\end{aligned}$$

# CHAPTER 8

# CONCLUSION AND OPEN PROBLEMS

In this thesis we have provided the first known non-trivial bounds on expansions and cut-sizes for graphs as a function of the hyperbolicity measure $\delta$, and have shown how these bounds and their related proof techniques lead to improved algorithms for two related combinatorial problems. We hope that these results will stimulate further research in characterizing the computational complexities of related combinatorial problems over asymptotic ranges of $\delta$. In addition to the usual future research of improving our bounds, the following interesting research questions remain:

- Can one use Lemma 14 or similar results to get a polynomial-time solution of UGC for some asymptotic ranges of $\delta$ An obvious recursive application using the approach in (80) encounters a hurdle since hyperbolicity is not a hereditary property (cf. Section 2.2), i.e., removal of nodes or edges may change $\delta$ sharply; however, it is conceivable that a more clever approach may succeed.
- Can our bounds on expansions and cut-sizes be used to get an improved approximation for the multicut problem [(112), Problem 18.1] provided $\delta = o(\log n)$?

Then we have attempted to formulate and analyze curvature analysis methods to provide the foundations of systematic approaches to find critical components and anomaly detection in networks by using geometric network curvatures.

Notions of curvatures of higher-dimensional geometric shapes and topological spaces play a *fundamental* role in physics and mathematics in characterizing anomalous behaviors of these higher dimensional entities. However, using curvature measures to detect anomalies in networks is *not* yet very common due to several reasons such as lack of preferred geometric interpretation of networks and lack of experimental evidences that may lead to specific desired curvature properties.

This research must *not* be viewed as uttering the final word on appropriateness and suitability of specific curvature measures, but rather should be viewed as a stimulator and motivator of further theoretical or empirical research on the exciting interplay between notions of curvatures from network and non-network domains.

There is a *plethora* of interesting future research questions and directions raised by the topical discussions and results in this regard. Some of these are stated below.

- ▷ For geometric curvatures, we considered the first-order non-trivial measure $\mathfrak{C}_d^2$. It would be of interest to investigate computational complexity issues of anomaly detection problems using $\mathfrak{C}_d^p$ for $p > 2$. We conjecture that our algorithmic results for extremal anomaly detection using $\mathfrak{C}_d^2$ (Theorem 15 ($a$2-2)&($b$2)) can be extended to $\mathfrak{C}_d^3$.

- ▷ There are at least two more aspects of geometric curvatures that need further careful investigation. Firstly, the topological association of elementary components to higher-dimensional objects by *no* which means the only reasonable topological association possible. But, more importantly, other suitable notions of geometric curvatures are quite possible. As a very simple illustration, assuming that smaller dimensional simplexes edges in the discrete network setting correspond to vectors or directions in the smooth context, an analogue of the Bochner-Weitzenböck formula developed by Forman for the curvature for a simplex $s$ can be given by the formula (32; 66):

$$\mathfrak{F}(s) = w_s \left( \left( \sum_{s \prec s'} \frac{w_s}{w_{s'}} + \sum_{s' \prec s} \frac{w_{s'}}{w_s} \right) - \sum_{s' \| s} \left| \sum_{s, s' \prec g} \frac{\sqrt{w_s w_{s'}}}{w_g} + \sum_{g \prec s, s'} \frac{w_g}{\sqrt{w_s w_{s'}}} \right| \right)$$

where $a \prec b$ means $a$ is a face of $b$, $a \parallel b$ means $a$ and $b$ have either a common higher-dimensional face or a common lower-dimensional face but *not* both, and $w$ is a function that assigns weights to simplexes. One can then either modify the Euler characteristics as $\sum_{k=0}^{p}(-1)^k\,\mathfrak{F}(f_d^k)$ or by combining the individual $\mathfrak{F}(f_d^k)$ values using curvature functions defined by Bloch (12).

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