

Global Stability of Financial Networks : Measures, Evaluations and Policy

Implications

BY

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THESIS

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I dedicate this thesis to my parents (P.Kaligounder and S.Prabhavathi), my son (S.L.Pargavan), my husband (K.Satish Kumar) and my sisters (Gowri Kaligounder and Shobila Kaligounder) for their unconditional love and support.

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SUMMARY

Threats on the stability of a financial system can severely affect the functioning of the entire economy. Thus considerable emphasis was placed on analyzing the cause and effects of such threats. Recent crisis in the global financial world has generated renewed interests in fragilities of global financial networks among economists and regulatory authorities. The financial crisis in the current and the past decade has shown that one important cause of instability in global markets was the so-called *financial contagion*, namely the instabilities or failures of individual components in the network spreads to otherwise healthier components, affecting the entire system. In the first part of the thesis we formalized the homogeneous banking network model of Nier et al. (78), its corresponding heterogeneous version, formalize the synchronous shock propagation procedure outlined in (78; 40). We defined appropriate stability measures and investigate the computational complexity of evaluating these measures for various network topologies and parameters of interest. We performed a comprehensive empirical evaluation over more than 700,000 combinations of network types and parameter combinations. Our results and proofs also shed some light on the properties of topologies and parameters of network that may lead to higher or lower stabilities.

In the second part of the thesis we consider a banking network model introduced by (91) A. Zawadoski. In his model the asset risks and counterparty risks are treated separately and each bank has only two counterparty neighbors, a bank fails due to the counterparty risk only if at least one of its two neighbors default. We consider the above model for more general network topologies, namely when each node has exactly $2r$ counterparty neighbors for some integer $r > 0$ and show that as the number of counterparty neighbors increase the probability of counterparty risk also increases, hence banks not only hedge their asset risk but also hedge its counterparty risk.

CHAPTER 1

INTRODUCTION

In market-based economies, financial systems perform important financial intermediation functions of borrowing from surplus units and lending to deficit units. Financial stability is the ability of the financial systems to absorb shocks and perform its key functions, even in stressful situations. Since financial institutions that are governed by borrowing, lending and participation in risky investments played a crucial role in the crisis, they have attracted a major part of the attention of economists; see (54) for a survey. Threats to the stability of a financial system may severely affect the functioning of the entire economy, and thus considerable emphasis is placed on analyzing the cause and effect of such threats. Recent crisis in the global financial world has generated renewed interests in fragilities of global networks among economists and regulatory authorities. It has clearly exposed potential weaknesses of the global economic system, renewing interests in the determination of fragilities of various segments of the global economy.

From the previous crisis it can be seen that, when a financial crisis begins there seems to be only two options

- Bail out the financial capitalist
- Suffer more financial crisis

Since the after effects of financial crisis is very costly, its important that we take steps to prevent the financial crisis in future. The severity of the previous crisis could have been reduced, if we had understood the interconnectedness of the financial system and acted when the North Rock(British Bank) went bankrupt. There is a need not only to understand what has happened and why but also to take preventive steps to avoid future crisis. Our model considers the interconnectedness of the financial institutions and the transfer of

shocks which may enable the authorities to be warned about any future crisis, so preventive steps can be taken in advanced.

All the financial institutions are interconnected, the greediness of few financial institutions will not only threaten their own stability but will affect the stability of the whole economy. When there is an unexpected shock, the investors who made risky investments expecting greater returns will collapse and transfer their loss to other financial institutions causing them to default. Our model shows how the failure of one or more financial institutions will affect the otherwise healthy financial institution leading to economic crisis. Our model is an extension of the financial network originally proposed by Nier et al. (78). We have formalized the shock propagation procedure outlined in (78; 40), defined appropriate stability measures and investigate the computational complexities of evaluating these measures for various network topologies and parameters of interest. Based on our evaluations, we discovered many interesting implications of stability measure. We have shown that some topological properties and parameter combinations may flag the network as a possible fragile network. Flagging a network as vulnerable does not necessarily imply that such is the case, but such a network may require further analysis based on other aspects(rumors and panics) of free market economics that cannot be modeled.

The concept of instability of a market-based financial system due to factors such as debt financing of investments can be traced back to earlier works of economists such as Irving Fisher (46) and John Keynes (61) during the 1930s Great Depression era. However, the exact causes of such instabilities have not yet been unanimously agreed upon. Economists such as Ekelund and Thornton (41) contend that a major reason for the recent financial crisis is the enactment of an act that removed several restrictions on mixing investment and consumer banking, whereas other economists such as Calabria disagree with such an assertion (24). Some economists such as Minsky have argued that such instabilities are systemic for many modern market-based economic systems (75).

In the first part of the thesis we investigate systemic instabilities of the banking networks, an important component of modern capitalist economies of many countries. The financial crisis in the current and past decade has shown that an important component of instability in global financial markets is the so-called *financial contagion*, namely the spreadings of instabilities or failures of *individual* components of the network to other, perhaps healthier, components. Motivated by the global economic crisis in the current and the past decade the natural question to investigate is:

- What is the true characterization of such instabilities of banking networks, *i.e.*,
 - Are such instabilities systemic, e.g, caused by a repeal of Glass-Steagall act with subsequent development of specific properties of banking networks that allowed a ripple effect (27)?
 - Or, are such instabilities caused just by a few banks that were “too big to fail” and/or “a few individually greedy executives” ?

To investigate these types of questions, one must first settle the following issues:

- What is the *precise* model of the banking network that is studied?
- How *exactly* failures of individual banks propagated through the network to other banks?
- What is an *appropriate stability measure* and what are the computational properties of such a measure?

As prior researchers such as Allen and Babus (5) pointed out, graph-theoretic concepts provide a conceptual framework within which various patterns of connections between banks can be described and analyzed in a meaningful way by modeling banking networks as a *directed* network in which nodes represent the banks and the links represent the direct exposures between banks. Such a network-based approach to studying financial systems is particularly important for assessing financial stability, and in capturing the externalities that the risk associated with a single or small group of institutions may create for the entire system. Conceptually, links between banks have two *opposing* effects on contagion:

- More interbank links increase the opportunity for spreading failures to other banks (50): when one region of the network suffers from a crisis, another region also incurs a loss because their claims on the troubled region fall in value and, if this spillover effect is strong enough, it can cause a crisis in adjacent regions.
- More interbank links provide banks with a form of *coinsurance* against uncertain liquidity flows (6), *i.e.*, banks can insure against the liquidity shocks by exchanging deposits through links in the network.

One other motivation is to investigate global stabilities of financial networks from the point of view of a regulatory agency (as was also the case, for example, in (54)). A regulatory agency with sufficient knowledge about a part of a global financial network is expected to periodically evaluate the stability of the network, and flag the network *ex ante* for further analysis if it fails some preliminary test or exceeds some minimum threshold of vulnerability. In this motivation, flagging a network as vulnerable does not necessarily imply that such is the case, but that such a network requires further analysis based on other aspects of free market economics that are not or simply cannot be modeled. While too many false positives may drain the finite resources of a regulatory agency for further analysis and investigation, this motivation assumes that vulnerability is too important an issue to be left for an *ex post facto* analysis.

Similar to prior research works such as (36; 40; 50; 54; 78; 72), in the first part of the thesis, we study the vulnerability of financial networks assuming the absence of government intervention as banks become insolvent. While this is an extreme worst-case situation, the main goal of such type of studies is to see if the network can survive a shock even under extreme situations. A further reason for not allowing any intervention is, unlike the case of public health issues such as controlling spread of epidemics, government intervention in a capitalist financial system is often not allowed or requires complex political and administrative operatives.

In the second part of the thesis we consider issues related to the financial risk management which is a critical component of maintaining economic stability. Hedging is a risk management option that protects the owner of an asset from loss. It is the process of shifting risks to the futures market. The risks in the market must be first identified in order to manage the risk. To identify the risk one must examine both the immediate risk (asset risk) as well as the risk due to indirect effects (counter-party risk). Though hedging will minimize overall profit when markets are moving positive, it also helps in reducing risk during undesirable market conditions. However, as the owner hedges his/her asset risk to protect against defaults, the owner also gets exposed to the counter-party risk. In (91) A. Zawadoski introduces a banking network model in which the asset and counter-party risks are treated separately, and showed that, under certain situations, banks do not manage counter-party risk in equilibrium. In his model, each bank has only two counter-party neighbors, a bank fails due to the counter-party risk only if *at least* one of its two neighbors default, and such a counter-party risk is an event with *low* probability. Informally, the author shows that the banks will hedge their asset risks by appropriate OTC contracts, and, though it may be socially optimal for banks to insure against counter-party risk, in equilibrium banks will *not* choose to insure this low probability event. The OTC contract not only creates a contagion but also creates externalities which undermines the incentives of the banks to avert contagion. The model uses short term debt to finance their real asset. The failure in this model is from the liability side, where the investors run on the banks when they do not trust the bank, *i.e.*, the investors do not roll over the debts of the banks. Hence the contagion can be avoided only by increasing the equity and *not* by providing liquidity.

In the second part of the thesis we consider a *more general* network topologies of the above model, namely when each node has exactly $2r$ counter-party neighbors for some integer $r > 0$. To summarize, we extend the analysis of (91) to show that as the number of counter-party neighbors increase the probability of counter-party risk also increases, and in particular, the socially optimal solution becomes privately sustainable when each bank hedges its risk to a sufficiently large number of other banks. The counter-party risk

can be hedged by holding more equity, buying default insurance on their counterparties or collateralizing OTC contracts. Since holding excess capital or collateralizing OTC contracts is a wasteful use of scarce capital (91), when the banks choose to hedge their counter-party risk they buy the default insurance on their counterparties. More precisely, our conclusions for the general case of $2r$ neighbors are as follows:

- All the banks will still decide to hedge their asset risks.
- If the number of counter-party neighbors is at least $n/2$, then all banks will decide to insure their counterparties, and socially optimal solution in case of two counterparties for each bank now becomes privately optimal solution.
- In the limit when the number of banks n in the network tend to ∞ , as the number of counter-party neighbors approach $n - 1$, failure of *very few* of its counter-party banks will not affect a bank.

CHAPTER 2

FORMAL MEASURES OF GLOBAL STABILITY OF BANKING NETWORKS

Threats to the stability of a financial system may severely affect the functioning of the entire economy, and thus considerable emphasis is placed on the analyzing the cause and effect of such threats. The financial crisis in the current and past decade has shown that one important cause of instability in global markets is the so-called *financial contagion*, namely the spreadings of instabilities or failures of *individual* components of the network to other, perhaps healthier, components. This leads to a natural question of whether the regulatory authorities could have predicted and perhaps mitigated the current economic crisis by effective computations of some stability measure of the banking networks. Motivated by such observations, we consider the problem of defining and evaluating stabilities of both homogeneous and heterogeneous banking networks against propagation of *synchronous idiosyncratic shocks* given to a subset of banks. We formalize the homogeneous banking network model of Nier *et al.* (78) and its corresponding heterogeneous version, formalize the synchronous shock propagation procedures outlined in (78; 40), define two appropriate stability measures and investigate the computational complexities of evaluating these measures for various network topologies and parameters of interest. Our results and proofs also shed some light on the properties of topologies and parameters of the network that may lead to higher or lower stabilities.

2.1 Related prior research works

Although there is a large amount of literature on stability of financial systems in general and banking systems in particular, much of the prior research is on the empirical side or applicable to small-size networks. Two main categories of prior researches can be summarized as follows. The particular model used here is the model of Nier *et al.* (78). As stated before, definition of a precise stability measure and analysis of its computational complexity issues for stability calculation were not provided for these models before. Due to

the large volume of prior research works, only selected subset of related prior research works are reviewed here, many other exciting research results are included in the cited literature section.

Network formation Babus (11) proposed a model in which banks form links with each other as an insurance mechanism to reduce the risk of contagion. In contrast, Castiglionesi and Navarro (28) studied decentralization of the network of banks that is optimal from the perspective of a social planner. In a setting in which banks invest on behalf of depositors and there are positive network externalities on the investment returns, fragility arises when “not sufficiently capitalized” banks gamble with depositors’ money. When the probability of bankruptcy is low, the decentralized solution well-approximates the first objective of Babus.

Contagion spread in networks Although ordinarily one would expect the risk of contagion to be larger in a highly interconnected banking system, some empirical simulations indicate that shocks have an *extremely complex* effect on the network stability in the sense that higher connectivity among banks may sometimes lead to *lower* risk of contagion.

Allen and Gale (6) studied how a banking system may respond to contagion when banks are connected under different network structures, and found that, in a setting where consumers have the liquidity preferences as introduced by Diamond and Dybvig (37) and have random liquidity needs, banks perfectly insure against liquidity fluctuations by exchanging interbank deposits, but the connections created by swapping deposits expose the *entire system* to contagion. Allen and Gale (5) concluded that incomplete networks are *more* prone to contagion than networks with maximum connectivity since better-connected networks are more resilient via transfer of a proportion of the losses in one bank’s portfolio to more banks through interbank agreements. Freixas *et al.* (47) explored the case of banks that face liquidity fluctuations due to the uncertainty about consumers withdrawing funds. Gai and Kapadia (50) argued that the higher is the connectivity among banks the more will be the contagion effect during crisis. Haldane (53) suggested that contagion should be measured based on the interconnectedness of each institution within the financial system. Liedorp *et al.* (70) investigated if interconnectedness in the interbank market is a channel through which banks affect

each other's riskiness, and argued that both large lending and borrowing shares in interbank markets increase the riskiness of banks active in the *dutch* banking market.

Dasgupta (34) explored how linkages between banks, represented by cross-holding of deposits, can be a source of contagious breakdowns by investigating how depositors, who receive a private signal about fundamentals of banks, may want to withdraw their deposits if they believe that enough other depositors will do the same. Lagunoff and Schreft (68) considered a model in which agents are linked in the sense that the return on an agent's portfolio depends on the portfolio allocations of other agents. Iazzetta and Manna (57) used network topology analysis on monthly data on deposits exchange to gain more insight into the way a liquidity crisis spreads. Nier *et al.* (78) explored the dependency of systemic risks on the structure of the banking system via network theoretic approach and the resilience of such a system to contagious defaults. Kleindorfer *et al.* (64) argued that network analysis can play a crucial role in understanding many important phenomena in finance. Corbo and Demange (33) explored that given the exogenous default of set of banks, the relationship of the structure of interbank connections to the contagion risk of defaults. Babus (10) studied how the trade-off between the benefits and the costs of being linked changes depending on the network structure, and observed that, when the network is maximal, liquidity can be redistributed in the system to make the risk of contagion minimal.

Acemoglu *et al.* (2) and Zawadowski (91) do investigate stability of financial networks, but differently from this study. Both Acemoglu *et al.* (2) and Zawadowski (91) consider two specific network topologies, namely the ring topology and the complete network topology, as opposed to a more general class of topologies in this study. Both Acemoglu *et al.* (2) and Zawadowski (91) considered only the effect of the shock propagation for *a few* discrete time steps, as opposed to the study done here; in the terminology of (36), this can be thought of as a "violent death" of the network as opposed to the "slow poisoning death" that is investigated here. The model and structure/terms of bilateral interbank agreements in (2), namely that banks lend to one another through debt contracts with contingency covenants, is quite different from the

one considered here. As a results, the conclusions in (2; 91) do not directly apply to this model and the corresponding simulation environment.

Attribute propagation models have been investigated in the past in other contexts such as influence maximization in social networks (60; 30; 29; 22), disease spreading in urban networks (44; ?; 43), and percolation models in physics and mathematics (87). However, the shock propagation model in considered here is very different from all these models. For example:

- Almost all of the other models include a trivial solution in which the attribute spreads to the entire network if each node is injected individually with the attribute. This is not the case with the shock propagation model.
- If shocking a subset of nodes makes x nodes in the network fail, then adding more nodes to this subset may not necessarily lead to the failure of x or more than x nodes of the network.
- The complexity of many previous attribute propagation models arises due to the presence of cycles in the graph. In contrast, the shock propagation model may be highly complex *even when the given network is acyclic*. Instead, a key component of the complexity arises due to two or more directed paths sharing a node.

2.2 Financial network model and stability measure

As prior researchers (5; 78; 40; 64) have commented:

“conceptual frameworks from the theory of weighted graphs with additional parameters may provide a powerful tool for analysis of banking network models”.

Several parametric graph-theoretic models, differing in the way edges are interpreted and additional parameters are used to characterize the contagion, have been used by prior researchers in finance and banking industry to study various research questions involving financial systems (36; 40; 54; 56; 78; 49; 90; 76; 8; 32; 88; 91). As noted by researchers in (78; 8):

“the modelling challenge in studying banking networks lies not so much in analyzing a model that is flexible enough to represent all types of insolvency cascades, but in studying a model that can mimic the empirical properties of these different types of networks”. Amini et al. in (8)[pg:2] and Nier et al. in (78)[pg: 2037]

The insolvency propagation model formalized and evaluated here using a mathematically precise abstraction is similar to or a generalization of the models in (36; 40; 54; 78; 56; 50; 72; 8) that represent *cascades* of cash-flow insolvencies. As (8) observes, over-the-counter (OTC) derivatives and similar markets are *prone to this type of cascades*. In such markets parties deal *directly* with one another rather than passing through an exchange, and thus each party is subject to the risk that the other party does not fulfill its payment obligations. The following example from (8) illustrates chains of such interactions:

“Consider two parties A and B, such that A has a receivable from party B upon the realization of some event. If B does not dispose enough liquid reserves, it will default on the payment. Now consider that B has entered an off-setting contract with another party C, hedging its exposure to the random event. If C is cash-flow solvent, then the payment will flow through the intermediary B and reach A. However, if C is cash-flow insolvent and defaults, then the intermediary B might become cash-flow insolvent if it depends on receivables from C to meet its payment obligations to A”. Amini et al. in (8) [Pg:37]

The length of such chains of interactions in some OTC markets, like the *credit default swap market*, is *significant* (31; 73), thereby *increasing* the probability of cascade of cash-flow insolvencies (5). As (78) observes, an insolvency propagation model such as the one studied here

“conceptualises the main characteristics of a financial system using network theory by relating the cascading behavior of financial networks both to the local properties of the nodes

and to the underlying topology of the network, allowing us to vary continuously the key parameters of the network”. *Nier et al in (78) [pg:2035]*

Although the cascading effect studied is of somewhat special and simplified nature, as noted by (54):

“This is a deliberate oversimplification, aimed at a clearer understanding of how an initial failure can propagate shocks throughout the system”. *Haldane in (54) 20 January 2011, Vol 496, Nature[pg:535]*

We used two types of networks for our study.

- **Homogeneous Networks** : Networks in which the assets are equally distributed among all the nodes(banks) in the network.
- **Heterogeneous Networks** : Networks in which the assets need not be equally distributed among all the nodes(banks) in the network.

2.2.1 Homogeneous Networks: Balance Sheets and Parameters for Banks

A precise abstraction of the model as outlined in (78) which builds up on the works of Eboli (40) is considered. The network is modeled by a weighted directed graph $G = (V, F)$ of n nodes and m directed edges, where each node $v \in V$ corresponds to a bank (Bank_v) and each directed edge $(v, v') \in F$ indicates that Bank_v has an agreement to lend money to $\text{Bank}_{v'}$. Let $\text{deg}_{\text{in}}(v)$ and $\text{deg}_{\text{out}}(v)$ denote the in-degree and the out-degree of node v . The model has the following parameters:

$$\mathcal{E} = \text{total external asset}, \quad \mathcal{I} = \text{total inter-bank exposure}, \quad A = \mathcal{I} + \mathcal{E} = \text{total asset}$$

$$[0, 1] \ni \gamma = \text{percentage of equity to asset}, \quad w = w(e) = \frac{\mathcal{I}}{m} = \text{weight of edge } e \in F, \quad \Phi = \text{severity of shock } (1 \geq \Phi > \gamma)$$

This model assumes that all the depositors are insured for their deposits, e.g in United States the Federal Deposit Insurance Corporation provides such an insurance up to a maximum level. Thus, **the parameters**

TABLE I. The balance sheet for a node $v \in V$ (i.e., for Bank _{v})

Assets	Liabilities
$\iota_v = \text{deg}_{\text{out}}(v) \times w = \text{interbank asset}$ $e_v = (b_v - \iota_v) + \frac{\mathcal{E} - \sum_{v \in V} (b_v - \iota_v)}{n} = (b_v - \iota_v) + \frac{\mathcal{E}}{n}$ $= \text{share of total external asset } \mathcal{E}$ $a_v = e_v + \iota_v = b_v + \frac{\mathcal{E}}{n} = \text{total asset}$	$b_v = \text{deg}_{\text{in}}(v) \times w = \text{interbank borrowing}$ $c_v = \gamma \times a_v = \text{net worth (equity)}$ $d_v = \text{customer deposits}$ $\ell_v = b_v + c_v + d_v = \text{total liability}$
$a_v = \ell_v$ (balance sheet equation)	

d_v for all v will be omitted in this model. Similarly, ℓ_v quantities (which depend on the d_v 's) are also only necessary in writing the balance sheet equation and will not be used subsequently.

Note that the homogeneous model is completely described by the 4-tuple of parameters $\langle G, \gamma, \mathcal{I}, \mathcal{E} \rangle$.

2.2.2 Balance Sheets and Parameters for Heterogeneous Networks

The heterogeneous version of the model is the same as its homogeneous counterpart as described above, except that the shares of interbank exposures and external assets for different banks may be different. Formally, the following modifications are done in the homogeneous model:

- $w(e) > 0$ denotes the weight of the edge $e \in \mathcal{E}$ along with the constraint that $\sum_{e \in F} w(e) = \mathcal{I}$.
- $\iota_v = \sum_{e \in (v, v') \in F} w(e)$, and $b_v = \sum_{e \in (v', v) \in F} w(e)$.
- $e_v = (b_v - \iota_v) + \sigma_v \times (\mathcal{E} - \sum_{v \in V} (b_v - \iota_v))$ for some $\sigma_v > 0$ along with the constraint $\sum_{v \in V} \sigma_v = 1$. Since $\sum_{v \in V} (b_v - \iota_v) = 0$, this gives $e_v = (b_v - \iota_v) + \sigma_v \mathcal{E}$. Consequently, a_v now equals $b_v + \sigma_v \mathcal{E}$.

Denoting the m -dimensional vector of $w(e)$'s by \mathbf{w} and the n -dimensional vector of σ_v 's by $\boldsymbol{\sigma}$, the heterogeneous model is completely described by the 6-tuple of parameters $\langle G, \gamma, \mathcal{I}, \mathcal{E}, \mathbf{w}, \boldsymbol{\sigma} \rangle$.

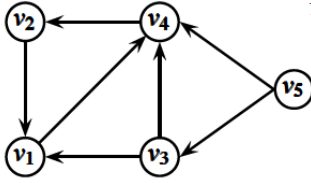


Figure 1. An example of the banking network model considered here.

n = number of nodes = 5

m = number of edges = 7

\mathcal{I} = total inter-bank exposure = $m = 7$

\mathcal{E} = total external asset = 14, $\gamma = 0.1$

Both homogeneous and heterogeneous network models are relevant in practice, and have been investigated by prior researchers such as (36; 54; 78; 56; 50; 72).

Illustration of calculations of balance sheet parameters

The calculation of relevant parameters of the balance sheet of banks for the simple banking network shown in Figure 1 is illustrated .

(a) Homogeneous version of the network

- w = weight of every edge = $\mathcal{I}/m = 1$.
- $\iota_{v_1} = \text{deg}_{\text{out}}(v_1) \times w = 1$, $\iota_{v_2} = \text{deg}_{\text{out}}(v_2) \times w = 1$, $\iota_{v_3} = \text{deg}_{\text{out}}(v_3) \times w = 2$, $\iota_{v_4} = \text{deg}_{\text{out}}(v_4) \times w = 1$, $\iota_{v_5} = \text{deg}_{\text{out}}(v_5) \times w = 2$.
- $b_{v_1} = \text{deg}_{\text{in}}(v_1) \times w = 2$, $b_{v_2} = \text{deg}_{\text{in}}(v_2) \times w = 1$, $b_{v_3} = \text{deg}_{\text{in}}(v_3) \times w = 1$, $b_{v_4} = \text{deg}_{\text{in}}(v_4) \times w = 3$, $b_{v_5} = \text{deg}_{\text{in}}(v_5) \times w = 0$.
- $e_{v_1} = b_{v_1} - \iota_{v_1} + \frac{\mathcal{E}}{n} = 3.8$, $e_{v_2} = b_{v_2} - \iota_{v_2} + \frac{\mathcal{E}}{n} = 2.8$, $e_{v_3} = b_{v_3} - \iota_{v_3} + \frac{\mathcal{E}}{n} = 1.8$, $e_{v_4} = b_{v_4} - \iota_{v_4} + \frac{\mathcal{E}}{n} = 4.8$, $e_{v_5} = b_{v_5} - \iota_{v_5} + \frac{\mathcal{E}}{n} = 0.8$.
- $a_{v_1} = b_{v_1} + \frac{\mathcal{E}}{n} = 4.8$, $a_{v_2} = b_{v_2} + \frac{\mathcal{E}}{n} = 3.8$, $a_{v_3} = b_{v_3} + \frac{\mathcal{E}}{n} = 3.8$, $a_{v_4} = b_{v_4} + \frac{\mathcal{E}}{n} = 5.8$, $a_{v_5} = b_{v_5} + \frac{\mathcal{E}}{n} = 2.8$.
- $c_{v_1} = \gamma a_{v_1} = 0.48$, $c_{v_2} = \gamma a_{v_2} = 0.38$, $c_{v_3} = \gamma a_{v_3} = 0.38$, $c_{v_4} = \gamma a_{v_4} = 0.58$, $c_{v_5} = \gamma a_{v_5} = 0.28$.

(b) Heterogeneous version of the network

Suppose that 95% of \mathcal{E} is distributed equally on the two banks v_1 and v_2 , and the rest 5% of \mathcal{E} is distributed equally on the remaining three banks.

Thus:

$$\sigma_{v_1}\mathcal{E} = \frac{0.95\mathcal{E}}{2} = 6.65, \sigma_{v_2}\mathcal{E} = \frac{0.95\mathcal{E}}{2} = 6.65, \sigma_{v_3}\mathcal{E} = \frac{0.05\mathcal{E}}{3} \approx 0.233, \sigma_{v_4}\mathcal{E} = \frac{0.05\mathcal{E}}{3} \approx 0.233, \sigma_{v_5}\mathcal{E} = \frac{0.05\mathcal{E}}{3} \approx 0.233$$

Suppose that 95% of \mathcal{S} is distributed equally on the three edges $f_1 = (v_2, v_1), f_2 = (v_1, v_4), f_3 = (v_4, v_2)$, and the remaining 5% of \mathcal{S} is distributed equally on the remaining four edges $f_4 = (v_3, v_1), f_5 = (v_3, v_4), f_6 = (v_5, v_4), f_7 = (v_5, v_3)$. Then,

$$w(f_1) = w(f_2) = w(f_3) = \frac{0.95\mathcal{S}}{3} \approx 2.216, w(f_4) = w(f_5) = w(f_6) = w(f_7) = \frac{0.05\mathcal{S}}{4} = 0.08725$$

$$\text{for bank } v_1: b_{v_1} = w(f_1) + w(f_4) \approx 2.30325, \iota_{v_1} = w(f_2) = 2.216$$

$$e_{v_1} = (b_{v_1} - \iota_{v_1}) + \sigma_{v_1}\mathcal{E} \approx 6.7365, a_{v_1} = b_{v_1} + \sigma_{v_1}\mathcal{E} \approx 8.9525, c_{v_1} = \gamma a_{v_1} \approx 0.8925$$

$$\text{for bank } v_2: b_{v_2} = w(f_3) \approx 2.216, \iota_{v_2} = w(f_1) \approx 2.216$$

$$e_{v_2} = (b_{v_2} - \iota_{v_2}) + \sigma_{v_2}\mathcal{E} = 6.65, a_{v_2} = b_{v_2} + \sigma_{v_2}\mathcal{E} \approx 8.866, c_{v_2} = \gamma a_{v_2} \approx 0.8666$$

$$\text{for bank } v_3: b_{v_3} = w(f_7) = 0.08725, \iota_{v_3} = w(f_4) + w(f_5) = 0.1745$$

$$e_{v_3} = (b_{v_3} - \iota_{v_3}) + \sigma_{v_3}\mathcal{E} \approx 0.14575, a_{v_3} = b_{v_3} + \sigma_{v_3}\mathcal{E} \approx 0.32025, c_{v_3} = \gamma a_{v_3} \approx 0.032035$$

$$\text{for bank } v_4: b_{v_4} = w(f_2) + w(f_5) + w(f_6) \approx 2.39050, \iota_{v_4} = w(f_3) \approx 2.216$$

$$e_{v_4} = (b_{v_4} - \iota_{v_4}) + \sigma_{v_4}\mathcal{E} \approx 0.4075, a_{v_4} = b_{v_4} + \sigma_{v_4}\mathcal{E} \approx 2.6235, c_{v_4} = \gamma a_{v_4} \approx 0.26235$$

$$\text{for bank } v_5: b_{v_5} = 0, \iota_{v_5} = w(f_6) + w(f_7) = 0.1745$$

$$e_{v_5} = (b_{v_5} - \iota_{v_5}) + \sigma_{v_5}\mathcal{E} \approx 0.0585, a_{v_5} = b_{v_5} + \sigma_{v_5}\mathcal{E} \approx 0.233, c_{v_5} = \gamma a_{v_5} \approx 0.0233$$

2.2.3 Initial insolvency via shocks

As in (78), the initial failures are caused by *idiosyncratic shocks* which can occur due to *operations risks* (frauds) or *credit risks*, and has the effect of reducing the external assets of a selected subset of banks

perhaps causing them to default. While *aggregated* or *correlated* shocks affecting all banks simultaneously is relevant in practice, idiosyncratic shocks are a cleaner way to study the *stability* of the topology of the banking network. Formally, a non-empty subset of nodes (banks) $\emptyset \subset V_{\text{shock}} \subseteq V$ is selected. For all nodes $v \in V_{\text{shock}}$, their external assets are simultaneously decreased from e_v by $s_v = \Phi e_v$, where the parameter $\Phi \in (0, 1]$ determines the “severity” of the shock. As a result, the new net worth of Bank_v becomes $c'_v = c_v - s_v$. The effect of this shock is as follows:

- If $c'_v \geq 0$, Bank_v continues to operate but with a lower net worth of c'_v .
- If $c'_v < 0$, Bank_v *defaults* (i.e., stops functioning).

2.2.4 Insolvency propagation equation

Let the notation $\text{deg}_{\text{in}}(v)$ denote the in-degree of node v . The insolvencies propagate in *discrete* time units $t = 0, 1, 2, \dots$; “ $(\dots, t, V_{\text{shock}}, \dots)$ ” is added to all relevant variables to indicate their dependences on t and on the set V_{shock} of initially shocked nodes. A bank becomes insolvent if its modified net worth becomes *negative*, and such a bank is *removed* from the network in the next time step. Let $V_{\text{ins}}(t, V_{\text{shock}}) \subseteq V$ denote the set of nodes that became insolvent *before* time t when an initial shock is provided to the nodes in V_{shock} (Thus, in particular, $\text{deg}_{\text{in}}(v, t, V_{\text{shock}})$ is the in-degree in the graph induced by the nodes in $V \setminus V_{\text{ins}}(t, V_{\text{shock}})$). The insolvencies of banks at time t affect the equity of other banks in the network at the next time step $t + 1$ by the following *non-linear* “insolvency propagation equation”

- An equation of same flavor with some simplification and omitted details was described in words by Nier *et al.* (78) and Eboli (40).
- Equation (Equation 2.1) is highly non-linear. The results in (36) indicate that *in general* it is NP-hard to find a subset V_{shock} of initially shocked nodes such that $\left| \lim_{t \rightarrow \infty} V_{\text{ins}}(t, V_{\text{shock}}) \right|$ is exactly or approximately maximized.

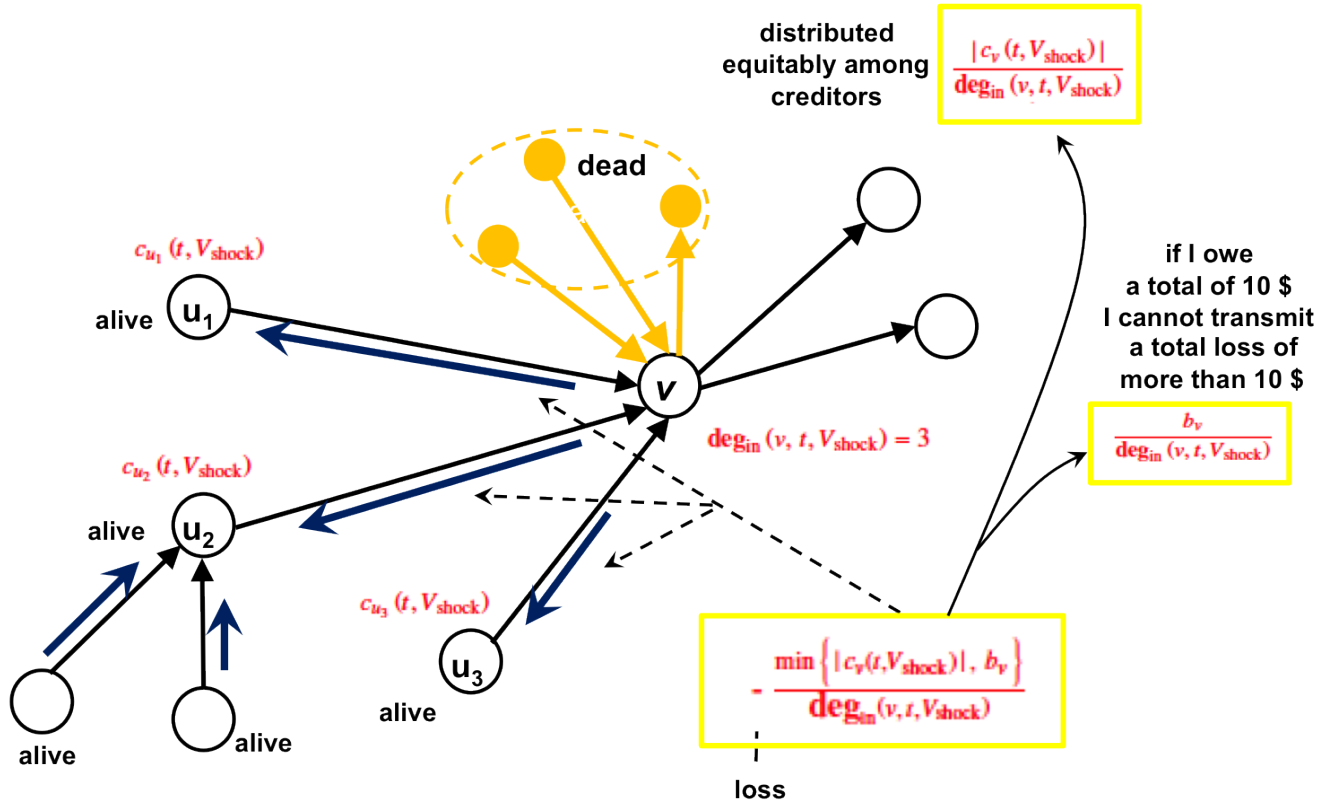


Figure 2. Pictorial illustration of the shock transmission equation for a node v from time t to time $t + 1$.

$$\forall u \in V \setminus V_{\text{ins}}(t, V_{\text{shock}}) : c_u(t+1, V_{\text{shock}}) = c_u(t, V_{\text{shock}}) - \sum_{\substack{(c_v(t, V_{\text{shock}}) < 0) \\ v : \wedge (v \in V \setminus V_{\text{ins}}(t, V_{\text{shock}})) \\ \wedge ((u, v) \in E)}} \frac{\min\{|c_v(t, V_{\text{shock}})|, b_v\}}{\text{deg}_{\text{in}}(v, t, V_{\text{shock}})} \quad (2.1)$$

In Equation (Equation 2.1), the term $\frac{|c_v(t, V_{\text{shock}})|}{\text{deg}_{\text{in}}(v, t, V_{\text{shock}})}$ ensures the loss of equity of an insolvent bank to be distributed equitably among its creditors that have not become insolvent yet, whereas the term $\frac{b_v}{\text{deg}_{\text{in}}(v, t, V_{\text{shock}})}$ ensures that the total loss propagated cannot be more than the total interbank exposure of the insolvent bank; see Figure 2 for a pictorial illustration. The insolvency propagation continues until *no* new bank becomes insolvent.

2.3 The Stability and Dual Stability Indices

A banking network is called *dead* if all the banks in the network have failed. Consider a given homogeneous or heterogeneous banking network $\langle G, \gamma, \mathcal{I}, \mathcal{E}, \Phi \rangle$ or $\langle G, \gamma, \mathcal{I}, \mathcal{E}, \Phi, \mathbf{w}, \boldsymbol{\sigma} \rangle$. For $\emptyset \subset V' \subseteq V$, let

$$\text{infl}(V') = \{v \in V \mid v \text{ fails if all nodes in } V' \text{ are shocked}\}$$

$$\text{SI}(G, V', T) = \begin{cases} |V'|/n, & \text{if } \text{infl}(V') = V \\ \infty, & \text{otherwise} \end{cases}$$

The Stability Index The optimal *stability index* of a network G is defined as

$$\text{SI}^*(G, T) = \text{SI}(G, V_{\text{shock}}, T) = \min_{V'} \{ \text{SI}(G, V', T) \}$$

For estimation of this measure, it is assumed that it is possible for the network to fail, *i.e.*, $\text{SI}^*(G, T) < \infty$. Thus, $0 < \text{SI}^*(G, T) \leq 1$, and the higher the stability index is, the better is the stability of the network against an idiosyncratic shock. The natural computational problem $\text{STAB}_{T,\phi}$ is thus considered. An optimal subset of nodes that is a solution of Problem $\text{STAB}_{T,\phi}$ by V_{shock} , *i.e.*, $\text{SI}^*(G, T) = \text{SI}(G, V_{\text{shock}}, T)$ is denoted. Note that if $T \geq n$ then the $\text{STAB}_{T,\phi}$ finds a minimum subset of nodes which, when shocked, will *eventually* cause the death of the network in an arbitrary number of time steps.

The Dual Stability Index Many covering-type minimization problems in combinatorics have a natural maximization dual in which one fixes a-priori the number of covering sets and then finds a maximum number of elements that can be covered with these many sets. For example, the usual dual of the minimum set covering problem is the maximum coverage problem (62). Analogously, a dual stability problem $\text{DUAL-STAB}_{T,\phi,\kappa}$ is defined. The *dual stability index* of a network G can then be defined as

$$\text{DSI}^*(G, T, \kappa) = \max_{V' \subseteq V: |V'|=\kappa} |\text{infl}(V') / \kappa|$$

The dual stability measure is of particular interest when $SI^*(G, T) = \infty$, *i.e.*, the entire network cannot be made to fail. In this case, a natural goal is to find out if a significant portion of the nodes in the network can be failed by shocking a limited number of nodes of G ; this is captured by the definition of $DSI^*(G, T, \kappa)$.

Violent Death vs. Slow Poisoning In the results, two cases of death of a network is distinguished:

violent death ($T = 2$) The network is dead by the very next step after the shock.

slow poisoning (any $T \geq 2$) The network may not be dead immediately but dies *eventually*.

<p>Input: a banking network with shocking parameter Φ, and an integer $T > 1$</p> <p>Valid solution: A subset $V' \subseteq V$ such that $SI(G, V', T) < \infty$</p> <p>Objective: <i>minimize</i> V'</p>	<p>Input: a banking network with shocking parameter Φ, and two integers $T, \kappa > 1$</p> <p>Valid solution: A subset $V' \subseteq V$ such that $V' = \kappa$</p> <p>Objective: <i>maximize</i> $\text{infl}(V')/\kappa$</p>
Stability of banking network ($STAB_{T,\phi}$)	Dual Stability of banking network ($DUAL-STAB_{T,\phi,\kappa}$)

2.4 Rationale Behind the Stability Measures

Although it is possible to think of many other alternate measures of stability for networks than the ones defined here, the measures introduced here are in tune with the ideas that references (78; 40) directly (and, some other references such as (49; 90) implicitly) used to empirically study their networks by shocking only a few (sometimes one) node. Thus, a rationale in defining the stability measures in the above manner is to follow the cue provided by other researchers in the banking industry in studying models such as the one discussed here instead of creating a completely new measure that may be out of sync with ideas used by prior researchers and therefore could be subject to criticisms.

2.5 Implications of our theoretical results on stability of banking networks

2.5.1 Effects of Topological Connectivity

Though researchers agree that the connectivity of banking networks affects its stability (6; 50), the conclusions drawn are mixed, namely some researchers conclude that lesser connectivity implies more sus-

ceptibility to contagion whereas other researchers conclude in the opposite. The results and their proofs, shows that topological connectivity does play a significant role in stability of the network in the following complex manner.

Even acyclic networks display complex stability behavior: Sometimes a cause of the instability of a banking network is attributed to *cyclical* dependencies of borrowing and lending mechanisms among major banks, e.g., banks v_1 , v_2 and v_3 borrowing from banks v_2 , v_3 and v_1 , respectively. The results here show that computing the stability measures may be difficult even without the presence of such cycles. Indeed, larger inapproximability results, especially for heterogeneous networks, are possible because slight change in network parameters can cause a large change in the stability measure. On the other hand, acyclic small-degree rooted in-arborescence networks exhibit higher values of the stability measure, e.g., if the maximum in-degree of any node in a rooted in-arborescence is 5 and the shock parameter Φ is no more than twice the value of the percentage of equity to assets γ , then by Theorem 2.8.5 $SI^*(G, T) > 0.1$.

Intersection of borrowing chains may cause lower stability: A *borrowing chain* means a directed path from a node v_1 to another node v_2 , indicating that bank v_2 effectively borrowed from bank v_1 through a sequence of successive intermediaries. Now, assume that there is another directed path from v_1 to another node v_3 . Then, failure of v_2 and v_3 propagates the resulting shocks to v_1 and, if the shocks arrive at the same step, then the total shock received by bank v_1 is the addition of these two shocks, which in turn passes this “amplified” shock to other nodes in the network.

Based on these kinds of observations, it can be reasonably inferred that homogeneous networks with topologies more like a small-degree in-arborescence have higher stabilities, whereas networks of other types of topologies may have lower stabilities even if the topologies are acyclic. For example, as can be observed later, when $\deg_{in}^{max} = 3$, $\gamma = 0.1$ and $\Phi = 0.15$, we get $SI^*(G, T) > 0.22$ and the network cannot be put to death without shocking more than 22% of the nodes.

2.5.2 Effects of Ratio of External to Internal Assets (\mathcal{E}/\mathcal{I}) and percentage of equity to assets (γ) for Homogeneous Networks

As the relevant results and their proofs provided here show, lower values of \mathcal{E}/\mathcal{I} and γ may cause the network stability to be extremely sensitive with respect to variations of other parameters of a homogeneous network. For example, in the proof of Theorem 2.8.2 we have $\lim_{n \rightarrow \infty} \mathcal{E}/\mathcal{I} = \lim_{n \rightarrow \infty} \gamma = 0$, leading to variation of the stability index by a logarithmic factor; however, in the proof of Theorem 2.8.4 we have $\mathcal{E}/\mathcal{I} = 0.25$ and $\gamma = 0.23$ leading to much smaller variation of the stability index.

2.5.3 Homogeneous vs. Heterogeneous Networks

The results and proofs provided here show that heterogeneous networks of banks with diverse equities tend to exhibit wider fluctuations of the stability index with respect to parameters, e.g., Theorem 2.8.12 shows a polylogarithmic fluctuation even if the ratio \mathcal{E}/\mathcal{I} is large.

2.6 Comparison with Other Models for Attribute Propagation in Networks

Models for propagation of beneficial or harmful attributes have been investigated in the past in several other contexts such as influence maximization in social networks (60; 30; 29; 22), disease spreading in urban networks (44; ?; 43), percolation models in physics and mathematics (87) and other types of contagion spreads (18; 17). However, the model for shock propagation in

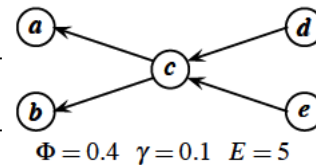


Figure 3. A homogeneous network used in the discussion in Section 2.6.

financial network discussed here is *fundamentally* very different from all these models. For example, the cascade models of failure considered in (18; 17) are probabilistic models of failure propagation of a more generic nature, and thus not very useful to study failure propagation via interlocked balance sheets of financial institutions (as is the case in OTC derivatives markets). Some distinguishing features of the model considered here include:

(a) Almost all of these models include a trivial solution in which the attribute spreads to the entire network if each node is injected individually with the attribute. This is not the case with the model considered here: *a node may not fail when shocked, and the network may not be dead if all nodes are shocked*. For example, consider the network in Figure 3.

- Suppose that all the nodes are shocked. Then, the following events happen.
 - Node a (and similarly node b) fails at $t = 1$ since $\Phi(\deg_{\text{in}}(a) + \frac{\epsilon}{5}) > \gamma(\deg_{\text{in}}(a) + \frac{\epsilon}{5})$.
 - Node c also fails at $t = 1$ since $\Phi(\deg_{\text{in}}(c) - \deg_{\text{out}}(c) + \frac{\epsilon}{5}) = 0.4 > \gamma(\deg_{\text{in}}(c) + \frac{\epsilon}{5}) = 0.3$.
 - Node d (and similarly node e) do not fail at $t = 1$ since $\Phi(-\deg_{\text{out}}(d) + \frac{\epsilon}{5}) = 0 < \gamma \times \frac{\epsilon}{5} = 0.1$ and its equity stays at $0.1 - 0 = 0.1$.
 - At $t = 2$, node d (and similarly node e) receives a shock from node c of the amount $\frac{0.4-0.3}{2} = 0.05 < 0.1$. Thus, nodes d and e do not fail. Since no new nodes fail during $t > 2$, the network does not become dead.
- However, suppose that only nodes a and b are shocked. Then, the following events happen.
 - Node a (and similarly node b) fails at $t = 1$ since $\Phi(\deg_{\text{in}}(a) + \frac{\epsilon}{5}) = 0.8 > \gamma(\deg_{\text{in}}(a) + \frac{\epsilon}{5}) = 0.2$.
 - At $t = 2$, node c receives a shock of the amount $2 \times (0.8 - 0.2) = 1.2 > \gamma(\deg_{\text{in}}(c) + \frac{\epsilon}{5}) = 0.3$. Thus, node c fails at $t = 2$.
 - At $t = 3$, node d (and similarly node e) receives a shock of the amount $\frac{1.2-0.3}{2} = 0.45 > \gamma \times \frac{\epsilon}{5} = 0.1$. Thus, both these nodes fail at $t = 3$ and the entire network is dead.

As the above example shows, if shocking a subset of nodes makes a network dead, adding more nodes to this subset may *not* necessarily lead to the death of the network, and the stability measure is *neither monotone nor sub-modular*. Similarly, it is also possible to exhibit banking networks such that to make the entire network fail:

- it may be necessary to shock a node even if it does not fail since shocking such a node “weakens” it by decreasing its equity, and
- it may be necessary to shock a node even if it fails due to shocks given to other nodes.

(b) The complexity of the computational aspects of many previous attribute propagation models arise due to the presence of cycles in the graph; for example, see (29) for polynomial-time solutions of some of these problems when the underlying graph does not have a cycle. In contrast, the computational problems considered here may be hard *even when the given graph is acyclic*; instead, a key component of computational complexity arises due to two or more directed paths sharing a node.

TABLE II. A summary of the results; $\varepsilon > 0$ is any arbitrary constant and $0 < \delta < 1$ is some constant.

	Network type, result type	Stability $SI^*(G, T)$ bound, assumption (if any), corresponding theorem	Dual Stability $DSI^*(G, T, \kappa)$ bound, assumption (if any), corresponding theorem
Homo- geneous	T=2 approximation hardness	$(1 - \varepsilon) \ln n$, $NP \not\subseteq DTIME(n^{\log \log n})$, Theorem 2.8.2	
	$T = 2$, approximation ratio	$O\left(\log\left(\frac{n \Phi \mathcal{E}}{\gamma (\Phi - \gamma) \mathcal{E} - \Phi }\right)\right)$, Theorem 2.8.3	
	Acyclic, $\forall T > 1$, approximation hardness	APX-hard, Theorem 2.8.4	$(1 - e^1 + \varepsilon)^{-1}$, $P \neq NP$, Theorem 2.8.14(a)
	In-arborescence, $\forall T > 1$, exact solution	$O(n^2)$ time, every node fails when shocked, Theorem 2.8.5	$O(n^3)$ time, every node fails when shocked, Theorem 2.8.14(b)
Hetero- geneous	Acyclic, $\forall T > 1$, approximation hardness	$(1 - \varepsilon) \ln n$, $NP \not\subseteq DTIME(n^{\log \log n})$, Theorem 2.8.10	$(1 - e^1 + \varepsilon)^{-1}$, $P \neq NP$, Theorem 2.8.14(a)
	Acyclic, $T = 2$, approximation hardness		n^δ , assumption $(\star)^\dagger$, Theorem 2.8.15
	Acyclic, $\forall T > 3$, approximation hardness	$2^{\log^{1-\varepsilon} n}$, $NP \not\subseteq DTIME(n^{\text{poly}(\log n)})$, Theorem 2.8.12	
	Acyclic, $T = 2$, approximation ratio ‡	$O\left(\log\left(\frac{n \mathcal{E} \overline{w_{\max}} \overline{w_{\min}} \overline{\sigma_{\max}}}{\Phi \gamma (\Phi - \gamma) \mathcal{E} \underline{w_{\min}} \underline{\sigma_{\min}} \underline{w_{\max}}}\right)\right)$, Theorem 2.8.11	

‡ See Theorem 2.8.11 for definitions of some parameters in the approximation ratio.

† See page 67 for statement of assumption (\star) , which is weaker than the assumption $P \neq NP$.

2.7 Overview of Our Results and Their Implications on Banking Networks

Table II summarizes the results, where the notation $\text{poly}(x_1, x_2, \dots, x_k)$ denotes a constant-degree polynomial in variables x_1, x_2, \dots, x_k . The results for heterogeneous networks show that the problem of computing stability indices for them is harder than that for homogeneous networks, as one would naturally expect.

2.7.1 Brief Overview of Proof Techniques

2.7.1.1 Homogeneous Networks, $\text{STAB}_{T,\phi}$

$T = 2$, approximation hardness and approximation algorithm : The reduction for approximation hardness is from a corresponding inapproximability result for the dominating set problem for graphs. The logarithmic approximation *almost* matches the lower bound. Even though this algorithmic problem can be cast as a covering problem, one *cannot* explicitly enumerate *exponentially many* covering sets in polynomial time. Instead, The problem is reformulated to that of computing an optimal solution of a polynomial-size integer linear programming (ILP), and then use the greedy approach of (38) for approximation. A careful calculation of the size of the coefficients of the ILP ensures that we have the desired approximation bound.

Any $T > 1$, approximation hardness and exact algorithm : The APX-hardness result, which holds even if the degrees of all nodes are *small* constants, is via a reduction from the node cover problem for 3-regular graphs. Technical complications in the reduction arise from making sure that the generated graph instance of $\text{STAB}_{T,\phi}$ is *acyclic*, no new nodes fail for any $t > 3$, but the network can be dead without each node being individually shocked. If the network is a rooted in-arborescence and every node can be individually shocked to fail, then an $O(n^2)$ time *exact* algorithm via dynamic programming was designed; as a by product it also follows that the value of the stability index of this kind of network with *bounded* node degrees is *large*.

2.7.1.2 Homogeneous Networks, $\text{DUAL-STAB}_{T,\phi,\kappa}$

Any T , approximation hardness and exact algorithm : For hardness, a lower bound for the *maximum coverage* problem (45) was translated. The reduction relies on the fact that in dual stability measure every

node of the network need *not* fail. If the given graph is a rooted in-arborescence and every node can be individually shocked to fail, an $O(n^3)$ time exact algorithm via dynamic programming is provided.

2.7.1.3 Heterogeneous Networks, $\mathbf{STAB}_{T,\phi}$

Any T , approximation hardness : The reduction is from a corresponding inapproximability result for the minimum set covering problem. Unlike homogeneous networks, unequal shares of the total external assets by various banks allows us to encode an instance of set cover by “equalizing” effects of nodes.

$T = 2$: The **approximation algorithm** uses linear program in Theorem 2.8.3 with more careful calculations.

Any $T > 2$, approximation hardness : This stronger poly-logarithmic inapproximability result than that in Theorem 2.8.10 is obtained by a reduction from MINREP, a graph-theoretic abstraction of two prover multi-round protocol for any problem in NP. Many technical complications in the reduction, culminating to a set of 22 symbolic linear equations between the parameters must be satisfied. Intuitively, the two provers in MINREP correspond to two nodes in the network that cooperate to fail to another specified set of nodes.

2.7.1.4 Heterogeneous Networks, $\mathbf{DUAL-STAB}_{2,\phi,\kappa}$,

approximation hardness : The reduction for this stronger inapproximability result is from the *densest hyper-graph* problem.

2.8 Detailed proofs of bounds in Table II

Proposition 2.8.1. *Let $\langle G = (V, F), \gamma, \beta, \mathcal{E} \rangle$ be the given (homogeneous or heterogeneous) banking network.*

Then, the following are true:

(a) *If $\deg_{\text{out}}(v) = 0$ for some $v \in V$, then node v must be given a shock (and, must fail due to this shock) for the entire network to fail.*

(b) *Let α be the number of edges in the longest directed simple path in G . Then, no new node fails at any time $t > \alpha$.*

(c) It can be assumed without loss of generality that G is weakly connected, i.e., the un-oriented version of G is connected.

Proof.

(a) Since $\deg_{\text{out}}(v) = 0$, no part of any shock given to any other nodes in the network can reach v . Thus, the network of v , namely $c_v = \gamma a_v$ stays strictly positive (since $\gamma > 0$) and node v never fails.

(b) Let t_{last} be the latest time a node of G failed, and let $V(t)$ be the set of nodes that failed at time $t = 1, 2, \dots, t_{\text{last}}$. Then, $V(1), V(2), \dots, V(t_{\text{last}})$ is a partition of V . For every $i = 1, 2, \dots, t_{\text{last}} - 1$, add directed edges (u, v) from a node $u \in V(i)$ to a node $v \in V(i + 1)$ if u was last node that transmitted any part of the shock to v before v failed. Note that (u, v) is also an edge of G and for every node $v \in V(i + 1)$ there must be an edge (u, v) for some node $u \in V(i)$. Thus, G has a path of length at least t_{last} .

(c) This holds since otherwise the stability measures can be computed separately on each weakly connected component. □

2.8.1 Homogeneous Networks, $\text{STAB}_{2,\phi}$, Logarithmic Inapproximability

Theorem 2.8.2. $\text{SI}^*(G, 2)$ cannot be approximated in polynomial time within a factor of $(1 - \varepsilon) \ln n$, for any constant $\varepsilon > 0$, unless $\text{NP} \subseteq \text{DTIME}(n^{\log \log n})$.

Proof. The *dominating set* problem for an undirected graph (DOMIN-SET) is defined as follows: given an undirected graph $G = (V, F)$ with $n = |V|$ nodes, find a minimum cardinality subset of nodes $V' \subset V$ such that every node in $V \setminus V'$ is incident on at least one edge whose other end-point is in V' . It is known that DOMIN-SAT is equivalent to the minimum set-cover problem under L-reduction (13), and thus cannot be approximated within a factor of $(1 - \varepsilon) \ln n$ unless $\text{NP} \subseteq \text{DTIME}(n^{\log \log n})$ (45).

Consider an instance $G = (V, F)$ of DOMIN-SET with n nodes and m edges, and let OPT denote the size of an optimal solution for this instance. The (directed) banking network $\vec{G} = (\vec{V}, \vec{F})$ is obtained from G by replacing each undirected edge $\{u, v\}$ by two directed edges (u, v) and (v, u) . Thus we have

$0 < \deg_{\text{in}}(v) = \deg_{\text{out}}(v) < n$ for every node $v \in V$. The global parameters are set as follows: $\mathcal{E} = 10n$, $\gamma = n^{-2}$ and $\Phi = 1$.

For a node v , let $\text{Nbr}(v) = \{u \mid \{u, v\} \in \mathcal{E}\}$ be the set of neighbors of v in G . It can be claimed that if a node v is shocked at time $t = 1$, then all nodes in $\{v\} \cup \text{Nbr}(v)$ fail at time $t = 2$. Indeed, suppose that v is shocked at $t = 1$. Then, v surely fails because

$$\Phi e_v = \deg_{\text{in}}(v) - \deg_{\text{out}}(v) + \frac{\mathcal{E}}{n} = 10 > \frac{2}{n} > \frac{\deg_{\text{in}}(v) + \frac{\mathcal{E}}{n}}{n^2} = \gamma a_v$$

Now, consider $t = 2$ and consider a node v such that v has not failed but a node $u \in \text{Nbr}(v)$ failed at time $t = 1$. Then, node v surely fails because

$$\begin{aligned} s_{v,2} &\geq \frac{\min\{s_{u,1} - c_u, b_u\}}{\deg_{\text{in}}(u, 2)} = \frac{\min\{\Phi e_u - \gamma a_u, \deg_{\text{in}}(u)\}}{\deg_{\text{in}}(u)} > \min\left\{\frac{10 - \frac{2}{n}}{\deg_{\text{in}}(u)}, 1\right\} \\ &> \frac{2}{n} > \frac{\deg_{\text{in}}(v) + \frac{\mathcal{E}}{n}}{n^2} = \gamma a_v \end{aligned}$$

Thus, we have a 1–1 correspondence between the solutions of DOMIN-SET and death of \vec{G} , namely $V' \subset V$ is a solution of DOMIN-SET if and only if shocking the nodes in V' makes \vec{G} fail at time $t = 2$. \square

2.8.2 Homogeneous Networks, $\text{STAB}_{2,\phi}$, Logarithmic Approximation

Theorem 2.8.3. *$\text{STAB}_{2,\phi}$ admits a polynomial-time algorithm with approximation ratio $O\left(\log\left(\frac{n \Phi \mathcal{E}}{\gamma (\Phi - \gamma) |\mathcal{E} - \Phi|}\right)\right)$.*

Proof. Suppose that $\Phi e_u < 0$ for some node $u \in V$. Then, there exists an optimal solution in which the node u is not shocked. Indeed, if u was shocked, the equity of u increases from c_u to $c_u + |\Phi e_u|$ and u does not propagate any shock to other nodes. Thus, if u still fails at $t = 2$, then it also fails at $t = 2$ if it was not shocked.

Let V_{shock} denote the set of nodes that are selected for shocking, and, for every node $v \in V$, let $\delta_{v,u}$ be defined as:

$$\delta_{v,u} = \begin{cases} \max\{0, \Phi e_v\}, & \text{if } u = v \\ \frac{\min\{\Phi e_v - c_v, b_v\}}{\deg_{\text{in}}(v)}, & \text{if } \Phi e_v > c_v \text{ and } (u, v) \in F \\ 0, & \text{otherwise} \end{cases}$$

Then, the problem reduces to a covering problem of the following type:

find a minimum cardinality subset $V_{\text{shock}} \subseteq V$ such that, for every node u , $\sum_{v \in V_{\text{shock}}} \delta_{v,u} > c_u$.

Note that it cannot even be explicitly enumerated, for a node $u \in V$, all subsets $V' \subseteq V \setminus \{u\}$ such that $\sum_{v \in V'} \delta_{v,u} > c_u$, since there are exponentially many such subsets. Let the binary variable $x_v \in \{0, 1\}$ be the indicator variable for a node $v \in V$ for inclusion in V_{shock} . However, the problem can be reformulated as the following integer linear programming problem:

$$\begin{aligned} & \text{minimize } \sum_{v \in V} x_v \\ & \text{subject to } \forall u \in V: \sum_{v \in V} \delta_{v,u} x_v > c_u \\ & x_u \in \{0, 1\} \end{aligned} \tag{2.2}$$

Let $\zeta = \min_{u \in V} \{ \min_{v \in V} \{\delta_{u,v}\}, c_u \}$. To ensure that every non-zero entry is at least 1 each constraint $\sum_{v \in V} \delta_{v,u} x_v > c_u$ can be rewritten as $\sum_{v \in V} \frac{\delta_{v,u}}{\zeta} x_v > \frac{c_u}{\zeta}$. Since the coefficients of the constraints and the objective function are all positive real numbers, (Equation 2.2) can be approximated by the greedy algorithm described in (38, Theorem 4.1) with an approximation ratio of $2 + \ln n + \ln \left(\max_{v \in V} \left\{ \sum_{u \in V} \frac{\delta_{v,u}}{\zeta} \right\} \right)$. Now, observe that:

$$\min_{\substack{u \in V \\ \delta_{u,u} > 0}} \{\delta_{u,u}\} = \min_{\substack{u \in V \\ \delta_{u,u} > 0}} \left\{ \Phi \left(\deg_{\text{in}}(u) - \deg_{\text{out}}(u) + \frac{\mathcal{E}}{n} \right) \right\} = \Omega \left(\frac{|\mathcal{E} - \Phi|}{n} \right)$$

$$\min_{u \in V} \min_{\substack{v \in V \\ \delta_{u,v} > 0}} \{\delta_{u,v}\} = \min_{u \in V} \min_{\substack{v \in V \\ \Phi e_v > c_v}} \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}}{\deg_{\text{in}}(v)} \right) - \Phi \frac{\deg_{\text{out}}(v)}{\deg_{\text{in}}(v)} \right\} = \Omega \left(\frac{(\Phi - \gamma) \mathcal{E}}{n} \right)$$

$$\begin{aligned} \min_{u \in V} \{c_u\} &= \min_{u \in V} \left\{ \gamma \left(\deg_{\text{in}}(u) + \frac{\mathcal{E}}{n} \right) \right\} = \Omega \left(\frac{\gamma \mathcal{E}}{n} \right) \\ \zeta &= \min \left\{ \min_{u \in V} \min_{v \in V} \{\delta_{u,v}\}, \min_{u \in V} \{c_u\} \right\} = \Omega \left(\min \left\{ \frac{|\mathcal{E} - \Phi|}{n}, \frac{(\Phi - \gamma) \mathcal{E}}{n}, \frac{\gamma \mathcal{E}}{n} \right\} \right) \\ \max_{v \in V} \sum_{u \in V} \delta_{v,u} &\leq n \max_{u \in V} \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}}{\deg_{\text{in}}(u)} \right) - \Phi \frac{\deg_{\text{out}}(u)}{\deg_{\text{in}}(u)} \right\} = O(n(\Phi - \gamma) \mathcal{E}) \end{aligned}$$

and thus, $\max_{v \in V} \left\{ \sum_{u \in V} \frac{\delta_{v,u}}{\zeta} \right\} = O \left(\text{poly} \left(n, \frac{\Phi}{\gamma}, \frac{1}{\Phi - \gamma}, \frac{\mathcal{E}}{|\mathcal{E} - \Phi|} \right) \right)$, giving the approximation bound. \square

2.8.3 Homogeneous Networks, $\text{STAB}_{T,\phi}$, any T , APX-hardness

Theorem 2.8.4. *For any T , computing $\text{Sl}^*(G, T)$ is APX-hard even if the banking network G is a DAG with $\deg_{\text{in}}(v) \leq 3$ and $\deg_{\text{out}}(v) \leq 2$ for every node v .*

Proof. The 3-MIN-NODE-COVER problem is reduced to $\text{STAB}_{T,\phi}$. 3-MIN-NODE-COVER is defined as follows. An undirected 3-regular graph G , i.e., an undirected graph $G = (V, F)$ in which the degree of every node is exactly 3 (and thus $|F| = 1.5|V|$) is given. A valid solution (node cover) is a subset of nodes $V' \subseteq V$ such that every edge is incident to at least one node in V' . The goal is then to find a node cover $V' \subseteq V$ such that $|V'|$ is *minimized*. This problem is known to be APX-hard (15).

Given such an instance $G = (V, F)$ of 3-MIN-NODE-COVER, an instance of the banking network $\vec{G} = (\vec{V}, \vec{F})$ is constructed as follows:

- For every node $v_i \in V$, we have two nodes u_i, u'_i in \vec{V} , and a directed edge (u_i, u'_i) . The node u'_i is referred as a “super-source” node.
- For every edge $\{v_i, v_j\} \in F$ with $i < j$, there is a (“sink”) node $e_{i,j}$ in \vec{V} and two directed edges $(e_{i,j}, u_i)$ and $(e_{i,j}, u_j)$ in \vec{F} . For notational convenience, the node $e_{i,j}$ is also sometimes referred to as the node $e_{j,i}$. Thus, $|\vec{V}| = 3.5|V|$, and $|\vec{F}| = 4|V|$. See Figure 4 for an illustration. Observe that:
- $\deg_{\text{in}}(u_i) = 3$ and $\deg_{\text{out}}(u_i) = 1$ for all $i = 1, 2, \dots, |V|$.

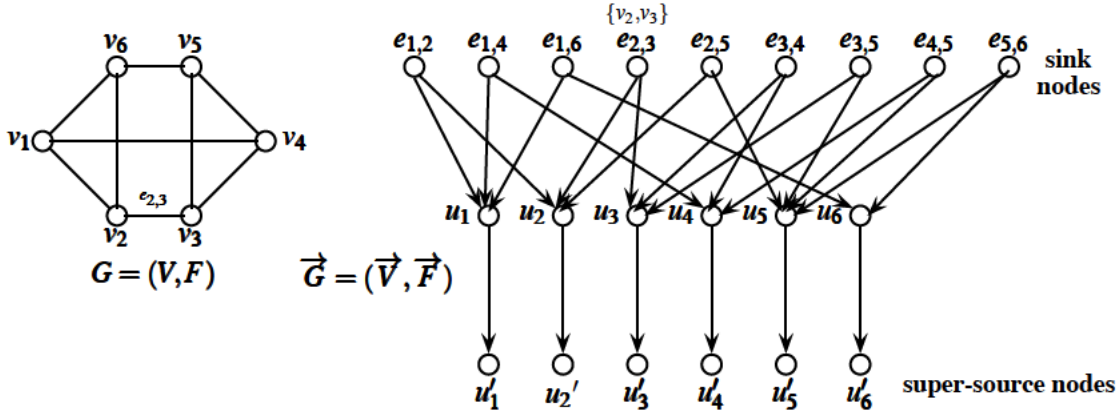


Figure 4. A 3-regular graph $G = (V, F)$ and its corresponding banking network $\vec{G} = (\vec{V}, \vec{F})$.

- $\deg_{\text{in}}(u_i') = 1$ and $\deg_{\text{out}}(u_i') = 0$ for all $i = 1, 2, \dots, |V|$. Thus, by Proposition 2.8.1(a), every node u_i' must be shocked to make the network fail.
- $\deg_{\text{in}}(e_{i,j}) = 0$ and $\deg_{\text{out}}(e_{i,j}) = 2$ for all i and j . Since $\deg_{\text{in}}(e_{i,j}) = 0$, if a node $e_{i,j}$ is shocked, no part of the shock is propagated to any other node in the network.
- Since the longest path in \vec{G} has 2 edges, by Proposition 2.8.1(b) no new node fails at any $t > 3$.

For notational convenience, let $n = |V|$, $E = \mathcal{E}/n$, and e_{i,j_1}, e_{i,j_2} and e_{i,j_3} be the three edges $\{v_i, v_{j_1}\}$, $\{v_i, v_{j_2}\}$ and $\{v_i, v_{j_3}\}$ in G that are incident on the node v_i . The remaining network parameters, namely γ , Φ and \mathcal{E} , are selected based on the following desirable properties.

(I) If a node u_i' is shocked at $t = 1$, it fails:

$$\Phi (\deg_{\text{in}}(u_i') - \deg_{\text{out}}(u_i') + E) > \gamma (\deg_{\text{in}}(u_i') + E) \equiv \Phi (1 + E) > \gamma (1 + E) \equiv \Phi > \gamma \quad (2.3)$$

(II) If a node $e_{i,j}$ is shocked, it does not fail:

$$\deg_{\text{in}}(e_{i,j}) - \deg_{\text{out}}(e_{i,j}) + E < 0 \equiv E < 2 \quad (2.4)$$

(III) If a node u_i is shocked at $t = 1$, then u_i fails at $t = 1$, and the nodes e_{i,j_1}, e_{i,j_2} and e_{i,j_3} fail at time $t = 2$ if they were not shocked (see Figure 5 for an illustration):

$$\begin{aligned} \frac{\min \{ \Phi(\deg_{\text{in}}(u_i) - \deg_{\text{out}}(u_i) + E) - \gamma(\deg_{\text{in}}(u_i) + E), \deg_{\text{in}}(u_i) \}}{\deg_{\text{in}}(u_i)} &> \gamma(\deg_{\text{in}}(e_{i,j_1}) + E) \\ &\equiv \frac{\min \{ \Phi(2 + E) - \gamma(3 + E), 3 \}}{3} > \gamma E \end{aligned}$$

The above inequality is satisfied provided:

$$\Phi(2 + E) > \gamma(3 + 4E) \quad (2.5)$$

$$1 > \gamma E \equiv \gamma < \frac{1}{E} \quad (2.6)$$

(IV) Consider a sink node $e_{i,j}$. Then, we required that if one or both of the super-source node u'_i and u'_j are

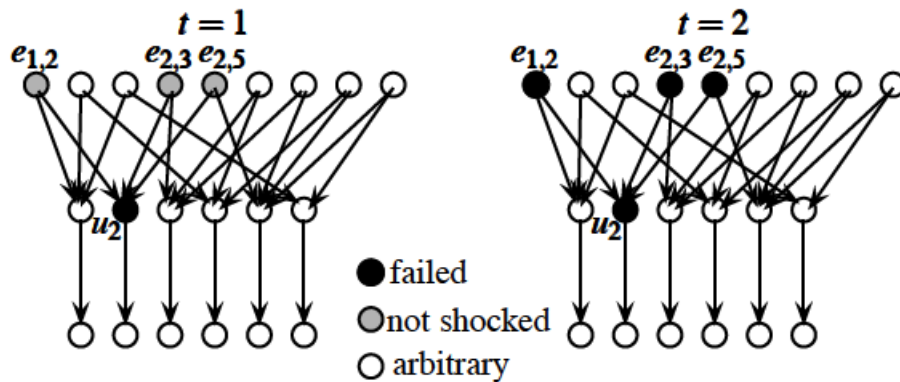


Figure 5. Case (III): if node u_2 is shocked then the nodes $e_{1,2}, e_{2,3}$ and $e_{2,5}$ must fail at $t = 2$.

shocked at $t = 1$ but the none of the nodes u_i , u_j and $e_{i,j}$ were shocked, then we require that one or both of the corresponding nodes u_i and u_j fail at $t = 2$, but the node $e_{i,j}$ never fails. Pictorially, we want a situation as depicted in Figure 6. This is satisfied provided the following inequalities hold:

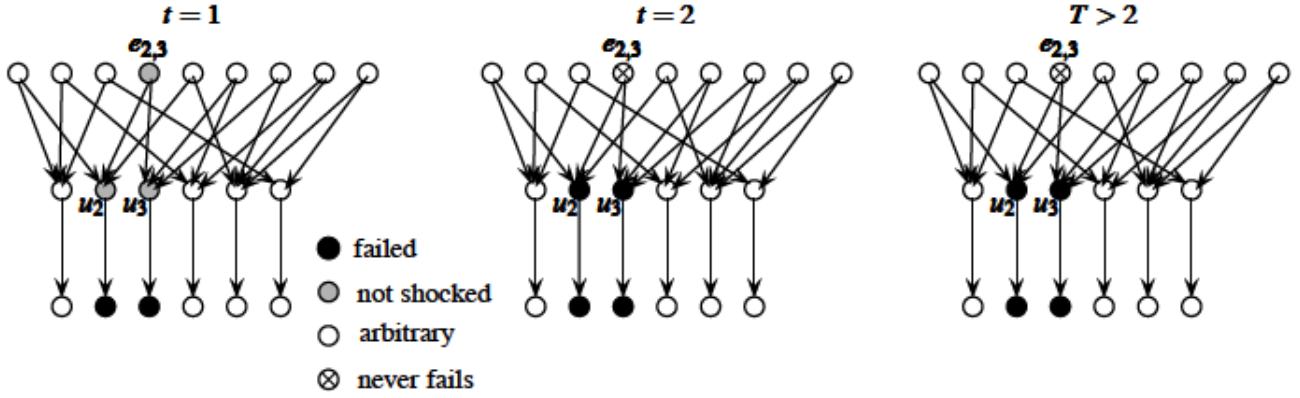


Figure 6. Case **(IV)**: to make $e_{2,3}$ fail, at least one of u_2 or u_3 must be shocked.

(IV-1) u_i fails at $t = 2$ if u'_i was shocked (the case of u_j and u'_j is similar):

$$\frac{\min \{ \Phi (\deg_{\text{in}}(u'_i) - \deg_{\text{out}}(u'_i) + E) - \gamma (\deg_{\text{in}}(u'_i) + E), \deg_{\text{in}}(u'_i) \}}{\deg_{\text{in}}(u'_i)} > \gamma (\deg_{\text{in}}(u_i) + E)$$

$$\equiv \frac{\min \{ (\Phi - \gamma)(1 + E), 1 \}}{1} > \gamma (3 + E)$$

The above inequality is satisfied provided:

$$(\Phi - \gamma)(1 + E) > \gamma (3 + E) \equiv \Phi(1 + E) > \gamma(4 + 2E) \quad (2.7)$$

$$1 > \gamma(3 + E) \equiv \gamma < \frac{1}{3 + E} \quad (2.8)$$

(IV-2) $e_{i,j}$ never fails even if both u_i and u_j have failed:

$$\frac{\min \{ (\Phi - \gamma)(1 + E), 1 \}}{1} - \gamma(3 + E) \leq \frac{\gamma E}{2} \equiv \min \{ (\Phi - \gamma)(1 + E), 1 \} \leq 3\gamma \left(1 + \frac{E}{2}\right)$$

The above inequality is satisfied provided:

$$(\Phi - \gamma)(1 + E) \leq 3\gamma \left(1 + \frac{E}{2}\right) \equiv \Phi(1 + E) \leq \gamma \left(4 + \frac{5E}{2}\right) \quad (2.9)$$

$$1 \leq 3\gamma \left(1 + \frac{E}{2}\right) \equiv \gamma \geq \frac{2}{6 + 3E} \quad (2.10)$$

There are obviously many choices of parameters γ , Φ and E that satisfy Equations (Equation 2.3)–(Equation 2.8); here just one is exhibited. Let $E = 1$ which satisfied Equation (Equation 2.4). Choosing $\gamma = 0.23$ satisfies Equations (Equation 2.6), (Equation 2.8) and (Equation 2.10). Letting $\Phi = 0.7$ satisfies Equations (Equation 2.3), (Equation 2.5), (Equation 2.7) and (Equation 2.9).

Suppose that $V' \subset V$ is a solution of 3-MIN-NODE-COVER. Then, all the super-nodes were shocked, and the nodes in V' . By **(I)** and **(III)** all the super-nodes and the nodes in $(\cup_{v_i \in V \setminus V'} \{v_i\})$ fails at $t = 1$, and by **(III)** the nodes in $\cup_{\substack{\{v_i, v_j\} \in \mathcal{E} \\ i < j}} \{e_{i,j}\}$ fails $t = 2$. Thus, a solution of \vec{G} by shocking $|V'| + n$ nodes was obtained.

Conversely, consider a solution of the $\text{STAB}_{T,\phi}$ problem on \vec{G} . Remember that all the super-nodes must be shocked, which ensures that $n + a$ nodes for some integer $a \geq 0$ needs to be shocked, and that any node v_i that is not shocked will fail at $t = 2$. By **(II)** it is of no use to shock the sink nodes. Thus, the shocked nodes consist of all super-nodes and a subset V' of cardinality a of the nodes u_1, u_2, \dots, u_n . By **(IV)** for every node $e_{i,j}$ at least one of the nodes u_i or u_j must be in U . Thus, the set of nodes $\{v_i \mid u_i \in U\}$ form a node cover of G of size a .

That the reduction is an L-reduction follows from the observation that any locally improvable solution of 3-MIN-NODE-COVER. has between $n/3$ and n nodes. \square

2.8.4 Restricted Homogeneous Networks, $\text{STAB}_{T,\phi}$, Any T , Exact Solution

The APX-hardness result of Theorem 2.8.4 has constant values for both Φ and γ , and requires $\deg_{\text{out}}(v) = 2$ for some nodes v . Here it is shown that if $\deg_{\text{out}}(v) \leq 1$ for every node v then under mild technical assumptions $\text{Sl}^*(G, T)$ can be computed in polynomial time for any T and, in addition, if $\deg_{\text{in}}(v)$ is bounded

by a constant for every node v then the network is highly stable (i.e., $\text{SI}^*(G, T)$ is large). Recall that an in-arborescence is a directed rooted tree where all edges are oriented towards the root.

Theorem 2.8.5. *If the banking network G is a rooted in-arborescence then $\text{SI}^*(G, T) > \frac{1}{1 + \deg_{\text{in}}^{\max} \left(\frac{\Phi}{\gamma} - 1 \right)}$, where $\deg_{\text{in}}^{\max} = \max_{v \in V} \{ \deg_{\text{in}}(v) \}$. Moreover, under the assumption that every node of G can be individually failed by shocking, $\text{SI}^*(G, T)$ can be computed exactly in $O(n^2)$ time.*

Remark 2.8.6. *Thus, for example, when $\deg_{\text{in}}^{\max} = 3$, $\gamma = 0.1$ and $\Phi = 0.15$, we get $\text{SI}^*(G, T) > 0.22$ and the network cannot be put to death without shocking more than 22% of the nodes. The proof gives an example for which the lower bound is tight.*

In the rest of this section, the above theorem is proved. Let $G = (V, F)$ be the given in-arborescence rooted at node r . The following notations and terminologies will be used:

- $u \rightarrow v$ and $u \rightsquigarrow v$ denote a directed edge and a directed path of one or more edges, respectively, from node u to node v .
- If $(u, v) \in F$ then v is the *parent* of u and u is a *child* of v . Similarly, if $u \rightsquigarrow v$ exists in G then v an *ancestor* of u and u a *descendent* of v .
- Let $\nabla(u) = \{ v \mid u \rightsquigarrow v \text{ exists in } G \}$ denote the set of all proper ancestors of u , and $\Delta(u) = \{ v \mid v \rightsquigarrow u \text{ exists in } G \} \cup \{ u \}$ denote the set of all descendants of u (including the node u itself). Note that for the network G to fail, at least one node in $\nabla(u) \cup \{ u \}$ must be shocked for every node u .

Suppose that a node u of G is shocked (and no other nodes in $\Delta(u)$ is shocked). If u fails, then the shock splits and propagates to a subset of nodes in $\Delta(u)$ until each split part of the shock terminates because of one of the following reasons:

- the component of the shock reaches a “leaf” node v with $\deg_{\text{in}}(v) = 0$, or
- the component of the shock reaches a node v with a sufficiently high c_v such that v does not fail.

Based on the above observations, the following quantities are defined .

Definition 2.8.7 (see Figure 7 for illustrations). *The influence zone of a shock on u , denoted by $iz(u)$, is the set of all failed nodes $v \in \Delta(u)$ within time T when u is shocked (and, no other node in $\Delta(u)$ is shocked).*

Note that $u \in iz(u)$.

Note that, for any node u , $iz(u)$ can be computed in $O(n)$ time.

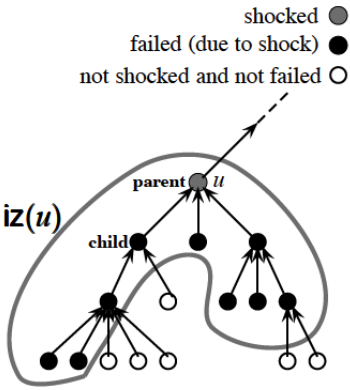


Figure 7. Influence zone of a shock on u

Lemma 2.8.8. *For any node u , $|iz(u)| < 1 + deg_{in}(u) \left(\frac{\Phi}{\gamma} - 1\right)$.*

Proof. For notational simplicity, let $E = \mathcal{E}/n$. If the node u does not fail when shocked, or u fails but it has no child, then $|z(u)| \leq 1$ and the claim holds since $\Phi > \gamma$. Otherwise, u fails and each of its $\deg_{\text{in}}(u)$ children at level 2 receives a part of the shock given by

$$\begin{aligned} \mathfrak{D} &= \min \left\{ \frac{\Phi(\deg_{\text{in}}(u) - 1 + E) - \gamma(\deg_{\text{in}}(u) + E)}{\deg_{\text{in}}(u)}, 1 \right\} \\ &< \Phi \left(1 + \frac{E}{\deg_{\text{in}}(u)} \right) - \gamma \left(1 + \frac{E}{\deg_{\text{in}}(u)} \right) \leq \Phi(1 + E) - \gamma(1 + E) \end{aligned}$$

Consider a child v of u . Each node $v' \in \Delta(v)$ that fails due to the shock subtracts an amount of $\gamma(\deg_{\text{in}}(v') + E) \geq \gamma(1 + E)$ from \mathfrak{D} provided this subtraction does not result in a negative value. Thus, the total number of failed nodes is strictly less than $1 + \deg_{\text{in}}(u) \frac{\Phi(1+E) - \gamma(1+E)}{\gamma(1+E)} = 1 + \deg_{\text{in}}(u) \left(\frac{\Phi}{\gamma} - 1 \right)$. \square

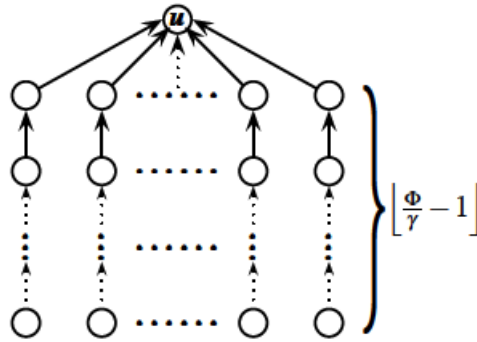


Figure 8. A tight example for the bound in Lemma 2.8.8 ($E = 0$).

Remark 2.8.9. *The bound in Lemma 2.8.8 is tight as shown in Figure 8.*

Lemma 2.8.8 immediately implies that

$$\text{Sl}^*(G, T) > \frac{\frac{n}{\max_{u \in V} \{iz(u)\}}}{n} > \frac{n / (1 + \deg_{\text{in}}^{\max}(\frac{\Phi}{\gamma} - 1))}{n} = \frac{1}{1 + \deg_{\text{in}}^{\max}(\frac{\Phi}{\gamma} - 1)}$$

A polynomial time algorithm to compute $\text{Sl}^*(G, T)$ exactly *assuming each node can be shocked to fail individually* is provided here. For a node u , define the following:

- For every node $u' \in \nabla(u)$, $\text{Sl}_{\text{SANS}}^*(G, T, u, u')$ is the number of nodes in an optimal solution of $\text{STAB}_{T, \phi}$ for the subgraph induced by the nodes in $\Delta(u)$ (or ∞ , if there is no feasible solution of $\text{STAB}_{T, \phi}$ for this subgraph under the stated conditions) assuming the following:
 - u' was shocked,
 - u was *not* shocked, and
 - no node in the path $u' \rightsquigarrow u$ excluding u' was shocked.
- $\text{Sl}_{\text{SAS}}^*(G, T, u)$ is the number of nodes in an optimal solution of $\text{STAB}_{T, \phi}$ for the subgraph induced by the nodes in $\Delta(u)$ (or ∞ , if there is no feasible solution of $\text{STAB}_{T, \phi}$ under the stated conditions)¹ assuming that the node u was shocked (and therefore failed).

The usual partition of the nodes of G into *levels*: $\text{level}(r) = 1$ and $\text{level}(u) = \text{level}(v) + 1$ is considered if u is a child of v . $\text{Sl}_{\text{SAS}}^*(G, T, u)$ and $\text{Sl}_{\text{SANS}}^*(G, T, u, v)$ for the nodes u are computed level by level, starting with the highest level and proceeding to successive lower levels. By Observation 2.8.1(a), the root r must be shocked to fail for the entire network to fail, and thus $\text{Sl}_{\text{SAS}}^*(G, T, r)$ will provide us with the required optimal solution.

¹Intuitively, a value of ∞ signifies that the corresponding quantity is undefined.

Every node u at the highest level has $\deg_{\text{in}}(u) = 0$. In general, $\text{SI}_{\text{SAS}}^*(G, T, u)$ and $\text{SI}_{\text{SANS}}^*(G, T, u, u')$ can be computed for any node u with $\deg_{\text{in}}(u) = 0$ as follows:

Computing $\text{SI}_{\text{SAS}}^*(G, T, u)$ when $\deg_{\text{in}}(u) = 0$: $\text{SI}_{\text{SAS}}^*(G, T, u) = 1$ by the assumption that every node can be shocked to fail.

Computing $\text{SI}_{\text{SANS}}^*(G, T, u, u')$ when $\deg_{\text{in}}(u) = 0$:

- If $u \in \text{iz}(u')$ then shocking node v makes node u fail. Since node u fails without being shocked, we have $\text{SI}_{\text{SANS}}^*(G, T, u, u') = 0$.
- Otherwise, node u does not fail. Thus, there is no feasible solution and $\text{SI}_{\text{SANS}}^*(G, T, u, u') = \infty$.

Note that only the number of nodes in $\Delta(u)$ is counted in the calculations of $\text{SI}_{\text{SANS}}^*(G, T, u, u')$ and $\text{SI}_{\text{SAS}}^*(G, T, u)$.

Now, consider a node u at some level ℓ with $\deg_{\text{in}}(u) > 0$. Let $v_1, v_2, \dots, v_{\deg_{\text{in}}(u)}$ be the children of u at level $\ell + 1$. Note that $\nabla(v_1) = \nabla(v_2) = \dots = \nabla(v_{\deg_{\text{in}}(u)})$.

Computing $\text{SI}_{\text{SAS}}^*(G, T, u)$ when $\deg_{\text{in}}(u) > 0$: By assumption, u fails when shocked. Note that no node in $\Delta(u) \setminus \{u\}$ can receive any component of a shock given to a node in $V \setminus \Delta(u)$ since u failed. For each child v_i of u there are two choices: v_i is shocked and (and, therefore, fails), or v_i is not shocked. Thus, in this case we have $\text{SI}_{\text{SAS}}^*(G, T, u) = 1 + \sum_{i=1}^{\deg_{\text{in}}(u)} \min \{ \text{SI}_{\text{SAS}}^*(G, T, v_i), \text{SI}_{\text{SANS}}^*(G, T, v_i, u) \}$.

Computing $\text{SI}_{\text{SANS}}^*(G, T, u, u')$ when $\deg_{\text{in}}(u) > 0$: Since u' is shocked and u is not shocked, the following cases arise:

- If $u \notin \text{iz}(u')$ then u does not fail. Thus, there is no feasible solution for the subgraph induced by the nodes in $\Delta(u)$ under this condition, and $\text{SI}_{\text{SANS}}^*(G, T, u, u') = \infty$.
- Otherwise, $u \in \text{iz}(u')$, and therefore u fails when u' is shocked. For each child v_i of u , there are two options: v_i is shocked and fails, or v_i is not shocked. Thus, in this case we have $\text{SI}_{\text{SANS}}^*(G, T, u, u') = \sum_{i=1}^{\deg_{\text{in}}(u)} \min \{ \text{SI}_{\text{SAS}}^*(G, T, v_i), \text{SI}_{\text{SANS}}^*(G, T, v_i, u') \}$.

```

(* preprocessing *)
 $\forall u \in V$ : compute  $\text{iz}(u)$ 
(* dynamic programming *)
for  $\ell = \ell_{\max}, \ell_{\max} - 1, \dots, 1$  do
  for each node  $u$  at level  $\ell$  do
    if  $\text{deg}_{\text{in}}(u) = 0$  then
       $\text{Sl}_{\text{SAS}}^*(G, T, u) = 1$ 
       $\forall u' \in \nabla(u)$ : if  $u \in \text{iz}(u')$  then  $\text{Sl}_{\text{SANS}}^*(G, T, u, u') = 0$  else  $\text{Sl}_{\text{SANS}}^a(G, T, u, u') = \infty$ 
    else (*  $\text{deg}_{\text{in}}(u) > 0$  *)
       $\text{Sl}_{\text{SAS}}^*(G, T, u) = 1 + \sum_{i=1}^{\text{deg}_{\text{in}}(u)} \min \{ \text{Sl}_{\text{SAS}}^*(G, T, v_i), \text{Sl}_{\text{SANS}}^*(G, T, v_i, u) \}$ 
       $\forall u' \in \nabla(u)$ : if  $u \notin \text{iz}(u')$  then  $\text{Sl}_{\text{SANS}}^*(G, T, u, u') = \infty$ 
      else
         $\text{Sl}_{\text{SANS}}^*(G, T, u, u') = \sum_{i=1}^{\text{deg}_{\text{in}}(u)} \min \{ \text{Sl}_{\text{SAS}}^*(G, T, v_i), \text{Sl}_{\text{SANS}}^*(G, T, v_i, u') \}$ 
      endif
    endif
  endfor
endfor
return  $\text{Sl}_{\text{SAS}}^*(G, T, r)$  as the solution

```

Figure 9. A polynomial time algorithm to compute $\text{Sl}^*(G, T)$ when G is a rooted in-arborescence and each node of G fails individually when shocked.

Let ℓ_{\max} be the maximum level number of any node in G . Based on the above observations, the dynamic programming algorithm as shown in Figure 9 to compute an optimal solution of $\text{STAB}_{T,\phi}$ on G can be designed. It is easy to check that the running time of the algorithm given here is $O(n^2)$.

2.8.5 Heterogeneous Networks, $\text{STAB}_{T,\phi}$, Any T , Logarithmic Inapproximability

Theorem 2.8.10. *Assuming $\text{NP} \not\subseteq \text{DTIME}(n^{\log \log n})$, for any constant $0 < \varepsilon < 1$ and any T , it is impossible to approximate $\text{Sl}^*(G, T)$ within a factor of $(1 - \varepsilon) \ln n$ in polynomial time even if G is a DAG.*

Proof. The (unweighted) SET-COVER problem is defined as follows. There is an universe \mathcal{U} of n elements, a collection of m sets \mathcal{S} over \mathcal{U} . The goal is to pick a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ containing a *minimum* number of sets such that these sets “cover” \mathcal{U} , i.e., $\cup_{S \in \mathcal{S}'} S = \mathcal{U}$. It is known that there exists instances of SET-COVER that cannot be approximated within a factor of $(1 - \delta) \ln n$, for any constant $0 < \delta < 1$, unless $\text{NP} \subseteq \text{DTIME}(n^{\log \log n})$ (45). Without any loss of generality, one may assume that every element $u \in \mathcal{U}$ belongs to at least two sets in \mathcal{S} since otherwise the only set containing u must be selected in any solution.

Given such an instance $\langle \mathcal{U}, \mathcal{S} \rangle$ of SET-COVER, an instance of the banking network $G = (V, F)$ is constructed as follows:

- There is a special node \mathfrak{B} .
- For every set $S \in \mathcal{S}$, there is a node S , and a directed edge (S, \mathfrak{B}) .
- For every element $u \in \mathcal{U}$, there is a node u , and directed edges (u, S) for every set S that contains u .

$\mathcal{U} = \{u_1, u_2, u_3, u_4\}$
 $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$
 $S_1 = \{u_1, u_2, u_3\}$
 $S_2 = \{u_3, u_4\}$
 $S_3 = \{u_3\}$
 $S_4 = \{u_1, u_2\}$

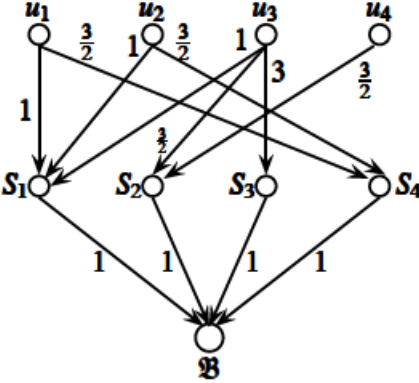


Figure 10. An instance $\langle \mathcal{U}, \mathcal{S} \rangle$ of SET-COVER and its corresponding banking network $G = (V, F)$.

Thus, $|V| = n + m + 1$, and $|F| < n m + m$. See Figure 10 for an illustration. The shares of internal assets for each bank is set as follows:

- For each set $S \in \mathcal{S}$, if S contains $k > 1$ elements then, for each element $u \in S$, The weight of the edge $e = (u, S)$ is set as $w(e) = \frac{3}{k}$.
- For each set $S \in \mathcal{S}$, the weight of the edge (S, \mathfrak{B}) is set as 1.

Thus, $\mathcal{I} = 4m$. Also, observe that:

- For any $S \in \mathcal{S}$, $b_S = 3$, and $\iota_S = 1$.
- For any $u \in \mathcal{U}$, $b_u = 0$. Also, since u belongs to at least two sets in \mathcal{S} and any set has at most $n - 1$ elements, $\frac{2}{n} \leq \iota_u < \frac{3n}{2}$.
- $b_{\mathfrak{B}} = m$ and $\iota_{\mathfrak{B}} = 0$.
- Since $\deg_{\text{in}}(u) = 0$ for any element $u \in \mathcal{U}$, if a node u is shocked, no part of the shock is propagated to any other node in the network.
- Since the longest path in G has 2 edges, by Proposition 2.8.1(b) no new node in G fails for $T > 3$.

Let the share of external assets for a node (bank) y be denoted by \mathcal{E}_y (thus, $\sum_{y \in V} \mathcal{E}_y = \mathcal{E}$). The remaining network parameters, namely γ , Φ and the \mathcal{E}_y values, will be selected based on the following properties.

(I) If the node \mathfrak{B} is shocked at $t = 1$, it fails:

$$\Phi(b_{\mathfrak{B}} - \iota_{\mathfrak{B}} + \mathcal{E}_{\mathfrak{B}}) > \gamma(b_{\mathfrak{B}} + \mathcal{E}_{\mathfrak{B}}) \equiv \Phi(m + \mathcal{E}_{\mathfrak{B}}) > \gamma(m + \mathcal{E}_{\mathfrak{B}}) \equiv \Phi > \gamma \quad (2.11)$$

(II) For any $S \in \mathcal{S}$, if node S is shocked at $t = 1$, then S fails at $t = 1$, and, for every $u \in S$, node u fails at time $t = 2$:

$$\begin{aligned} \frac{\min\{\Phi(b_S - \iota_S + \mathcal{E}_S) - \gamma(b_S + \mathcal{E}_S), b_S\}}{\deg_{\text{in}}(S)} &> \gamma(b_u + \mathcal{E}_u) \\ &\equiv \frac{\min\{\Phi(2 + \mathcal{E}_S) - \gamma(3 + \mathcal{E}_S), 3\}}{|S|} > \gamma \mathcal{E}_u \end{aligned}$$

The above inequality is satisfied if:

$$\Phi(2 + \mathcal{E}_S) > \gamma(3 + \mathcal{E}_S + |S| \mathcal{E}_u) \quad (2.12)$$

$$\Phi(2 + \mathcal{E}_S) - \gamma(3 + \mathcal{E}_S) \leq 3 \quad (2.13)$$

(III) For any $u \in \mathcal{U}$, consider the node u , and let $S_1, S_2, \dots, S_p \in \mathcal{S}$ be the p sets that contain u . Then, it is required that if the node \mathfrak{B} is shocked at $t = 1$ then \mathfrak{B} fails at $t = 1$, every node among the set of nodes $\{S_1, S_2, \dots, S_p\}$ that was not shocked at $t = 1$ fails at $t = 2$, but the node u does not fail if the none of the nodes u, S_1, S_2, \dots, S_p were shocked, This is satisfied provided the following inequalities hold:

(III-1) Any node among the set of nodes $\{S_1, S_2, \dots, S_p\}$ that was not shocked at $t = 1$ fails at $t = 2$. This is satisfies provided for any set $S \in \mathcal{S}$ the following holds:

$$\begin{aligned} \frac{\min \{ \Phi (b_{\mathfrak{B}} - \iota_{\mathfrak{B}} + \mathcal{E}_{\mathfrak{B}}) - \gamma (b_{\mathfrak{B}} + \mathcal{E}_{\mathfrak{B}}), b_{\mathfrak{B}} \}}{\deg_{\text{in}}(\mathfrak{B})} &> \gamma (b_S + \mathcal{E}_S) \\ &\equiv \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right), 1 \right\} > \gamma (3 + \mathcal{E}_S) \end{aligned}$$

The above inequality is satisfied provided:

$$(\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) > \gamma (3 + \mathcal{E}_S) \equiv \Phi \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) > \gamma \left(4 + \mathcal{E}_S + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) \quad (2.14)$$

$$1 > \gamma (3 + \mathcal{E}_S) \equiv \gamma < \frac{1}{3 + \mathcal{E}_S} \quad (2.15)$$

(III-2) u does not fail if the none of the nodes u, S_1, S_2, \dots, S_p were shocked:

$$\begin{aligned} \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right), 1 \right\} - \gamma (3 + \mathcal{E}_S) &\leq \frac{\gamma \mathcal{E}_u}{n} \\ &\equiv \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right), 1 \right\} \leq \gamma \left(3 + \mathcal{E}_S + \frac{\mathcal{E}_u}{n} \right) \end{aligned}$$

The above inequality is satisfied provided:

$$(\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) \leq \gamma \left(3 + \mathcal{E}_S + \frac{\mathcal{E}_u}{n} \right) \equiv \Phi \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) \leq \gamma \left(4 + \mathcal{E}_S + \frac{\mathcal{E}_{\mathfrak{B}}}{m} + \frac{\mathcal{E}_u}{n} \right) \quad (2.16)$$

$$(\Phi - \gamma) \left(1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m} \right) \leq 1 \equiv \gamma \geq \Phi - \frac{1}{1 + \frac{\mathcal{E}_{\mathfrak{B}}}{m}} \quad (2.17)$$

There are many choices of parameters γ , Φ and \mathcal{E}_y 's satisfying Equations (Equation 2.11)–(Equation 2.17); just one is exhibited here:

$$\forall S \in \mathcal{S}: \mathcal{E}_S = 0 \quad \mathcal{E}_{\mathfrak{B}} = 0 \quad \forall u \in \mathcal{U}: \mathcal{E}_u = \frac{1}{100n} \quad \gamma = 0.1 \quad \Phi = 0.4 + \frac{1}{n^{10000}}$$

Suppose that $\mathcal{S}' \subseteq \mathcal{S}$ is a solution of SET-COVER. Then, the node \mathfrak{B} and the nodes S are shocked for each $S \in \mathcal{S}'$. By (I) and (II) the node \mathfrak{B} and the nodes S for each $S \in \mathcal{S}'$ fails at $t = 1$, and by (II) the nodes u for every $u \in \mathcal{U}$ fails $t = 2$. Thus, a solution of G by shocking $|\mathcal{S}'| + 1$ nodes are obtained.

Conversely, consider a solution of the $\text{STAB}_{T,\phi}$ problem on G . If a node u for some $u \in \mathcal{U}$ was shocked, instead the node S for any set S that contains u can be shocked, which by (II) still fails all the nodes in the network and does not increase the number of shocked nodes. Thus, after such normalizations, it can be assumed that the shocked nodes consist of \mathfrak{B} and a subset $\mathcal{S}' \subseteq \mathcal{S}$ of nodes. By (II) and (III) for every node $u \in \mathcal{U}$ at least one set that contains u must be in \mathcal{S}' . Thus, the collection of sets in \mathcal{S}' form a cover of \mathcal{U} of size $|\mathcal{S}'|$. \square

2.8.6 Heterogeneous Networks, $\text{STAB}_{2,\phi}$, Logarithmic Approximation

For any positive real $x > 0$, let $\bar{x} = \max\{x, 1/x\}$ and $\underline{x} = \min\{x, 1/x\}$. Let $w_{\min} = \min_{e: w(e)>0} \{w(e)\}$, $w_{\max} = \max_e \{w(e)\}$, $\sigma_{\min} = \min_{v: \sigma_v>0} \{\sigma_v\}$, and $\sigma_{\max} = \max_v \{\sigma_v\}$.

Theorem 2.8.11. *$\text{STAB}_{T,\phi}$ admits a poly-time algorithm with approximation ratio*

$$O\left(\log \frac{n \bar{\mathcal{E}} \overline{w_{\max}} \overline{w_{\min}} \overline{\sigma_{\max}}}{\Phi \gamma (\Phi - \gamma) \underline{\mathcal{E}} \underline{w_{\min}} \underline{\sigma_{\min}} \underline{w_{\max}}}\right).$$

Proof. The proof of the corresponding approximation for homogeneous networks in Theorem 2.8.3 can be reused to obtain an approximation ratio of $2 + \ln n + \ln \left(\max_{v \in V} \left\{ \sum_{u \in V} \frac{\delta_{v,u}}{\zeta} \right\} \right)$, where $\zeta = \min_{u \in V} \{ \min_{v \in V} \{ \delta_{u,v} \}, c_u \}$, provided $\max_{v \in V} \left\{ \sum_{u \in V} \frac{\delta_{v,u}}{\zeta} \right\}$ is recalculated. Then,

$$\begin{aligned} \min_{\substack{u \in V \\ \delta_{u,u} > 0}} \{ \delta_{u,u} \} &= \min_{\substack{u \in V \\ \delta_{u,u} > 0}} \left\{ \Phi \left(\sum_{e=(v',u) \in F} w(e) - \sum_{e=(u,v') \in F} w(e) + \sigma_v \mathcal{E} \right) \right\} = \Omega \left(\text{poly} \left(s, \Phi, \underline{\mathcal{E}}, \underline{\sigma_{\min}} \right) \right) \\ \min_{u \in V} \min_{\substack{v \in V \\ \delta_{u,v} > 0}} \{ \delta_{u,v} \} &= \min_{u \in V} \min_{\substack{v \in V \\ \Phi_{e_v > c_v}}} \left\{ \left(\Phi - \gamma \right) \left(\frac{\sigma_v \mathcal{E}}{\sum_{e=(v',v) \in F} w(e)} \right) - \Phi \frac{\sum_{e=(v,v') \in F} w(e)}{\sum_{e=(v',v) \in F} w(e)} \right\} \\ &= \Omega \left(\text{poly} \left(n^{-1}, \Phi - \gamma, \Phi, \underline{\mathcal{E}}, \underline{w_{\max}}, \underline{w_{\min}}, \underline{\sigma_{\min}} \right) \right) \end{aligned}$$

$$\begin{aligned} \min_{u \in V} \{ c_u \} &= \min_{u \in V} \left\{ \gamma \left(\sum_{e=(v',u) \in F} w(e) + \sigma_u \mathcal{E} \right) \right\} = \Omega \left(\text{poly} \left(n^{-1}, \gamma, \underline{\mathcal{E}}, \underline{\sigma_{\min}}, \underline{w_{\min}} \right) \right) \\ \zeta &= \min \left\{ \min_{u \in V} \min_{v \in V} \{ \delta_{u,v} \}, \min_{u \in V} \{ c_u \} \right\} = \Omega \left(\text{poly} \left(n^{-1}, \Phi - \gamma, \Phi, \gamma, \underline{\mathcal{E}}, \underline{w_{\min}}, \underline{\sigma_{\min}}, \underline{w_{\max}} \right) \right) \\ \max_{v \in V} \sum_{u \in V} \delta_{v,u} &\leq n \max_{u \in V} \left\{ \left(\Phi - \gamma \right) \left(\frac{\sigma_v \mathcal{E}}{\sum_{e=(v',v) \in F} w(e)} \right) - \Phi \frac{\sum_{e=(v,v') \in F} w(e)}{\sum_{e=(v',v) \in F} w(e)} \right\} = O \left(\text{poly} \left(n, \overline{\mathcal{E}}, \overline{w_{\max}}, \overline{w_{\min}}, \overline{\sigma_{\max}} \right) \right) \end{aligned}$$

and thus,

$$\max_{v \in V} \left\{ \sum_{u \in V} \frac{\delta_{v,u}}{\zeta} \right\} = O \left(\text{poly} \left(n, \Phi^{-1}, \gamma^{-1}, (\Phi - \gamma)^{-1}, \overline{\mathcal{E}}, \underline{\mathcal{E}}^{-1}, \overline{w_{\max}}, \overline{w_{\min}}, \overline{\sigma_{\max}}, \underline{w_{\min}}^{-1}, \underline{\sigma_{\min}}^{-1}, \underline{w_{\max}}^{-1} \right) \right)$$

giving the desired approximation bound. \square

2.8.7 Heterogeneous Networks, $\text{STAB}_{2,\phi}$, $T > 3$, Poly-logarithmic Inapproximability

Theorem 2.8.12. *Assuming $\text{NP} \not\subseteq \text{DTIME}(n^{\text{poly}(\log n)})$, for any constant $0 < \varepsilon < 1$ and any $T > 3$, it is impossible to approximate $\text{Sl}^*(G, T)$ within a factor of $2^{\log^{1-\varepsilon} n}$ in polynomial time even if G is a DAG.*

Proof. The MINREP problem (with minor modifications from the original setup) is defined as follows. We are given a bipartite graph $G = (V^{\text{left}}, V^{\text{right}}, F)$ such that the degree of every node of G is at least 10, a partition of V^{left} into $\frac{|V^{\text{left}}|}{\sigma}$ equal-size subsets $V_1^{\text{left}}, V_2^{\text{left}}, \dots, V_\sigma^{\text{left}}$, and a partition of V^{right} into $\frac{|V^{\text{right}}|}{\beta}$ equal-size subsets $V_1^{\text{right}}, V_2^{\text{right}}, \dots, V_\beta^{\text{right}}$.

These partitions define a natural “bipartite super-graph” $G_{\text{super}} = (V_{\text{super}}, F_{\text{super}})$ in the following manner. G_{super} has a “super-node” for every V_i^{left} (for $i = 1, 2, \dots, \sigma$) and for every V_j^{right} (for $j = 1, 2, \dots, \beta$). There exists an “super-edge” $h_{i,j}$ between the super-node for V_i^{left} and the super-node for V_j^{right} if and only if there exists $u \in V_i^{\text{left}}$ and $v \in V_j^{\text{right}}$ such that $\{u, v\}$ is an edge of G . A pair of nodes u and v of G “witnesses” the super-edge $h_{i,j}$ of H provided u is in V_i^{left} , v is in V_j^{right} and the edge $\{u, v\}$ exists in G , and a set of nodes $V' \subseteq V$ of G witnesses a super-edge if and only if there exists at least one pair of nodes in S that witnesses the super-edge.

The goal of MINREP is to find $V_1 \subseteq V^{\text{left}}$ and $V_2 \subseteq V^{\text{right}}$ such that $V_1 \cup V_2$ witnesses every super-edge of H and the size of the solution, namely $|V_1| + |V_2|$, is *minimum*. For notational simplicity, let $n = |V^{\text{left}}| + |V^{\text{right}}|$. The following result is a consequence of Raz’s parallel repetition theorem (81; 66).

Theorem 2.8.13. (66) *Let L be any language in NP and $0 < \delta < 1$ be any constant. Then, there exists a reduction running in $n^{\text{poly}(\log n)}$ time that, given an input instance x of L , produces an instance of MINREP such that:*

- if $x \in L$ then MINREP has a solution of size $\alpha + \beta$;
- if $x \notin L$ then MINREP has a solution of size at least $(\alpha + \beta) \cdot 2^{\log^{1-\delta} n}$.

Thus, the above theorem provides a $2^{\log^{1-\delta} n}$ -inapproximability for MINREP under the complexity-theoretic assumption of $\text{NP} \not\subseteq \text{DTIME}(n^{\text{polylog}(n)})$.

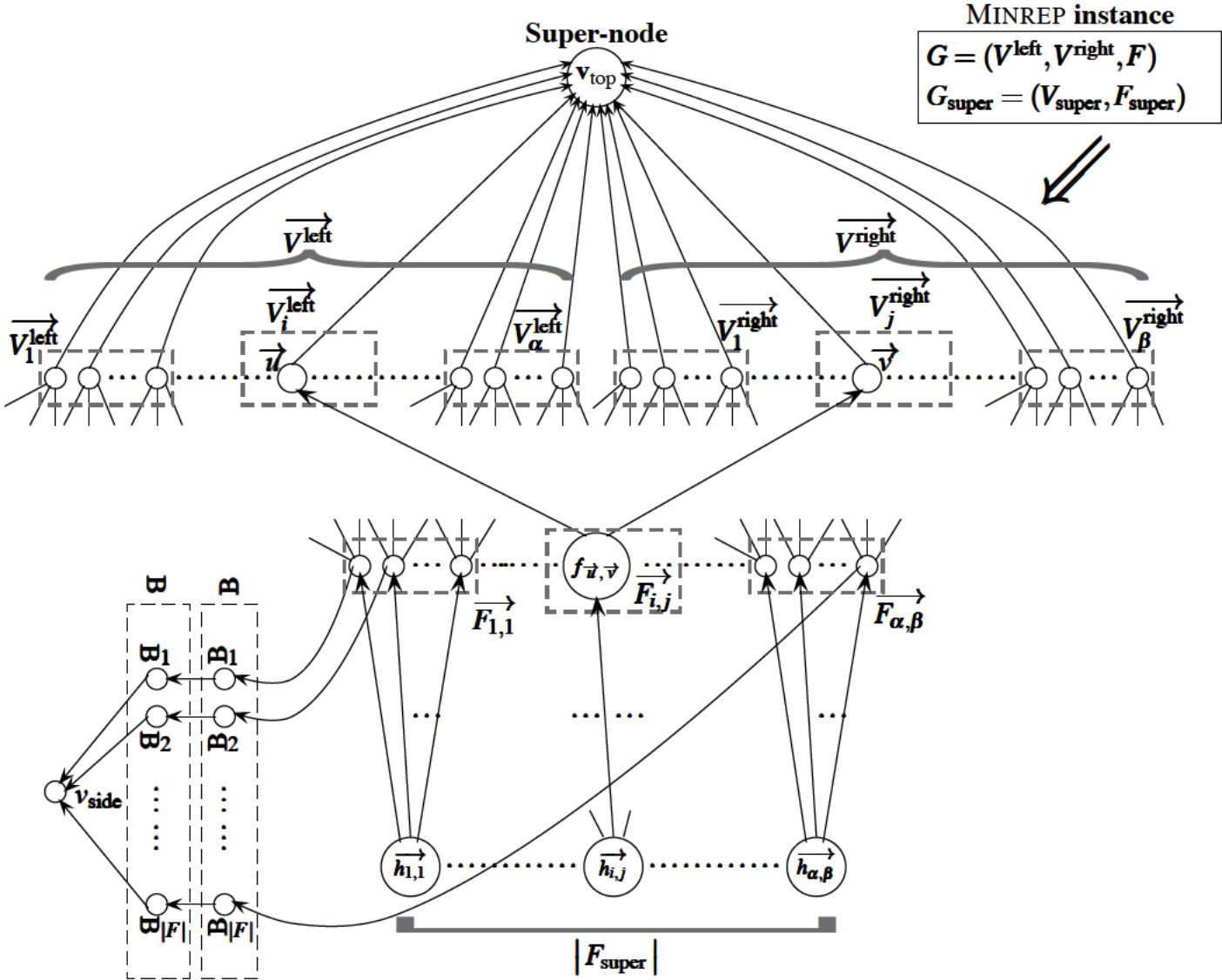


Figure 11. Reduction of an instance of MINREP to STAB_{T,φ} for heterogeneous networks.

Let $F_{i,j} = \{\{u, v\} \mid u \in V_i^{\text{left}}, v \in V_j^{\text{right}}, \{u, v\} \in F\}$. The construction of an instance of $\text{STAB}_{T,\phi}$ from an instance of MINREP is shown. The directed graph $\vec{G} = (\vec{V}, \vec{F})$ for $\text{STAB}_{T,\phi}$ is constructed as follows (see Figure 11 for an illustration):

Nodes:

- For every node $u \in V_i^{\text{left}}$ of G we have a corresponding node \vec{u} in the set of nodes \vec{V}_i^{left} in \vec{G} , and for every node $v \in V_j^{\text{right}}$ of G we have a corresponding node \vec{v} in the set of nodes \vec{V}_j^{right} in \vec{G} . The total number of such nodes is n .
- For every edge $\{u, v\}$ of G with $u \in V_i^{\text{left}}$ and $v \in V_j^{\text{right}}$, we have a corresponding node $f_{\vec{u}, \vec{v}}$ in the set of nodes $\vec{F}_{i,j}$ in \vec{G} . There are $|F|$ such nodes.
- For every super-edge $h_{i,j}$ of G_{super} , we have a node $\vec{h}_{i,j}$ in \vec{G} . There are $|F_{\text{super}}|$ such nodes.
- We have one “top super-node” v_{top} , one “side super-node” v_{side} , and $2|F|$ additional nodes $\varpi_1, \varpi_2, \dots, \varpi_{|F|}$, $\varpi_1, \varpi_2, \dots, \varpi_{|F|}$. Let $\varpi = \cup_{j=1}^{|F|} \varpi_j$ and $\varpi = \cup_{j=1}^{|F|} \varpi_j$.

Thus, $n + 3|F| + 2 < |\vec{V}| = n + |F| + |F_{\text{super}}| + 2 + 2|F| < n + 4|F| + 2$.

Edges:

- For every node u of G , we have an edge (u, v_{top}) in \vec{G} . There are n such edges.
- For every edge $\{u, v\}$ of G , we have two edges $(f_{\vec{u}, \vec{v}}, \vec{u})$ and $(f_{\vec{u}, \vec{v}}, \vec{v})$ in \vec{G} . There are $2|F|$ such edges.
- For every super-edge $h_{i,j}$ of G_{super} and for every edge $f_{u,v}$ in $F_{i,j}$, we have an edge $(\vec{h}_{i,j}, f_{\vec{u}, \vec{v}})$ in \vec{G} . There are $|F|$ such edges.
- Let $p_1, p_2, \dots, p_{|F|}$ be any arbitrary ordering of the edges in F . Then, for every $j = 1, 2, \dots, |F|$, we have the edges $(v_{\text{side}}, \varpi_j)$, (ϖ_j, ϖ_j) and (ϖ_j, p_j) . The total number of such edges is $3|F|$.

Thus, $|\vec{\mathcal{E}}| = n + 6|F|$.

Distribution of internal assets: The weight of every edge is set to 1, Thus, $\mathcal{S} = n + \sum_{u \in V^{\text{left}} \cup V^{\text{right}}} \deg(u) + 4|F| = n + 6|F|$.

Let $\deg(u) \geq 10$ be the degree of node $u \in V^{\text{left}} \cup V^{\text{right}}$. Observe that:

- $b_{v_{\text{top}}} = n$, and $\iota_{v_{\text{top}}} = 0$. Since $\deg_{\text{out}}(v_{\text{top}}) = 0$, by Proposition 2.8.1(a) the node v_{top} must be shocked to make the network fail.
- $b_{v_{\text{side}}} = |F|$, and $\iota_{v_{\text{side}}} = 0$. Since $\deg_{\text{out}}(v_{\text{side}}) = 0$, by Proposition 2.8.1(a) the node v_{side} must be shocked to make the network fail.
- For any $u \in V^{\text{left}} \cup V^{\text{right}}$, $b_{\vec{u}} = \deg(u)$ and $\iota_{\vec{u}} = 1$.
- For any node $f_{\vec{u}, \vec{v}}$, $b_{f_{\vec{u}, \vec{v}}} = 1$ and $\iota_{f_{\vec{u}, \vec{v}}} = 2$.
- For every node $\vec{h}_{i,j}$, $b_{\vec{h}_{i,j}} = 0$ and $\iota_{\vec{h}_{i,j}} = |F_{i,j}|$. Since $\deg_{\text{in}}(\vec{h}_{i,j}) = 0$ for any node $\vec{h}_{i,j}$, if such a node is shocked, no part of the shock is propagated to any other node in the network.
- For every j , $b_{\varpi_j} = \iota_{\varpi_j} = b_{\omega_j} = \iota_{\omega_j} = 1$.
- Since the longest directed path in G has 4 edges, by Proposition 2.8.1(b) no new node in G fails for $t > 4$.

Let the share of external assets for a node (bank) y be denoted by \mathcal{E}_y (thus, $\sum_{y \in V} \mathcal{E}_y = \mathcal{E}$). The remaining network parameters, namely γ , Φ and the set of \mathcal{E}_y values, are selected based on the following desirable properties and events. For the convenience of the readers, all the relevant constraints are also summarized in Table III. Assume that no nodes in $\left(\cup_{i,j} \vec{F}_{i,j}\right) \cup \left(\cup_{i,j} \left\{\vec{h}_{i,j}\right\}\right)$ were shocked at $t = 1$.

(I) Suppose that the node v_{top} is shocked at $t = 1$. Then, the following happens.

(I-a) v_{top} fails at $t = 1$:

$$\Phi(b_{v_{\text{top}}} - \iota_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}) > \gamma(b_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}) \equiv \Phi(n + \mathcal{E}_{v_{\text{top}}}) > \gamma(n + \mathcal{E}_{v_{\text{top}}}) \equiv \boxed{\Phi > \gamma} \quad (2.18)$$

(I-b) Each node $\vec{u} \in \overrightarrow{V^{\text{left}}} \cup \overrightarrow{V^{\text{right}}}$ that was not shocked at $t = 1$ fails at $t = 2$:

$$\begin{aligned} \frac{\min\{\Phi(b_{v_{\text{top}}} - \iota_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}) - \gamma(b_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}), b_{v_{\text{top}}}\}}{\deg_{\text{in}}(v_{\text{top}})} &> \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}}) \\ &\equiv \frac{\min\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\}}{n} > \gamma(\deg(u) + \mathcal{E}_{\vec{u}}) \end{aligned}$$

These constraints are satisfied provided:

$$\frac{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}})}{n} > \gamma(\deg(u) + \mathcal{E}_{\vec{u}}) \equiv \boxed{\Phi > \gamma \left(1 + \frac{\deg(u) + \mathcal{E}_{\vec{u}}}{1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}}\right)} \quad (2.19)$$

$$\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}) \leq n \equiv \boxed{\Phi \leq \gamma + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}}} \quad (2.20)$$

(I-c) If the nodes \vec{u}, \vec{v} and $f_{\vec{u}, \vec{v}}$ were *not* shocked at $t = 1$, then the part of the shock, say σ_1 , given

to v_{top} that is received by node $f_{\vec{u}, \vec{v}}$ at $t = 3$ is:

$$\begin{aligned} \sigma_1 &= \frac{\min\left\{\frac{\min\{\Phi(b_{v_{\text{top}}} - \iota_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}) - \gamma(b_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}), b_{v_{\text{top}}}\}}{\deg_{\text{in}}(v_{\text{top}})} - \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}}), b_{\vec{u}}\right\}}{\deg_{\text{in}}(\vec{u})} \\ &+ \frac{\min\left\{\frac{\min\{\Phi(b_{v_{\text{top}}} - \iota_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}) - \gamma(b_{v_{\text{top}}} + \mathcal{E}_{v_{\text{top}}}), b_{v_{\text{top}}}\}}{\deg_{\text{in}}(v_{\text{top}})} - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}}\right\}}{\deg_{\text{in}}(\vec{v})} \\ &= \frac{\min\left\{\frac{\min\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\}}{n} - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u)\right\}}{\deg(u)} \\ &+ \frac{\min\left\{\frac{\min\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\}}{n} - \gamma(\deg(v) + \mathcal{E}_{\vec{v}}), \deg(v)\right\}}{\deg(v)} \end{aligned}$$

On the other hand, if the node $f_{\vec{u},\vec{v}}$ and *exactly* one of the nodes \vec{u} and \vec{v} , say \vec{u} , were *not* shocked at $t = 1$, then the part of the shock, say σ'_1 , given to v_{top} that is received by node $f_{\vec{u},\vec{v}}$ at

$$t = 3 \text{ is: } \sigma'_1 = \frac{\min \left\{ \frac{\min \left\{ \Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n \right\}}{n} - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u) \right\}}{\deg(u)}$$

(II) Suppose that some node \vec{u} is shocked at $t = 1$. Then, the following happens.

(II-a) Node \vec{u} fails at $t = 1$:

$$\Phi(b_{\vec{u}} - \iota_{\vec{u}} + \mathcal{E}_{\vec{u}}) > \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}}) \equiv \boxed{\Phi > \gamma \left(1 + \frac{1}{\deg(u) - 1 + \mathcal{E}_{\vec{u}}} \right)} \quad (2.21)$$

(II-b) Node $f_{\vec{u},\vec{v}} \in \overrightarrow{F_{i,j}}$ fails at $t = 2$ and node $\overrightarrow{h_{i,j}}$ fails at $t = 3$ if both \vec{u} and \vec{v} were shocked at $t = 1$:

$$\begin{aligned} & \frac{\min \left\{ \Phi(b_{\vec{u}} - \iota_{\vec{u}} + \mathcal{E}_{\vec{u}}) - \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}}), b_{\vec{u}} \right\}}{\deg_{\text{in}}(\vec{u})} + \frac{\min \left\{ \Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}} \right\}}{\deg_{\text{in}}(\vec{v})} > \gamma(b_{f_{\vec{u},\vec{v}}} + \mathcal{E}_{f_{\vec{u},\vec{v}}}) \\ & \equiv \frac{\min \left\{ \Phi(\deg(u) - 1 + \mathcal{E}_{\vec{u}}) - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u) \right\}}{\deg(u)} \\ & \quad + \frac{\min \left\{ \Phi(\deg(v) - 1 + \mathcal{E}_{\vec{v}}) - \gamma(\deg(v) + \mathcal{E}_{\vec{v}}), \deg(v) \right\}}{\deg(v)} > \gamma(1 + \mathcal{E}_{f_{\vec{u},\vec{v}}}) \\ & \frac{\min \left\{ \frac{\min \left\{ \Phi(b_{\vec{u}} - \iota_{\vec{u}} + \mathcal{E}_{\vec{u}}) - \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}}), b_{\vec{u}} \right\}}{\deg_{\text{in}}(\vec{u})} + \frac{\min \left\{ \Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}} \right\}}{\deg_{\text{in}}(\vec{v})} - \gamma(b_{f_{\vec{u},\vec{v}}} + \mathcal{E}_{f_{\vec{u},\vec{v}}}), b_{f_{\vec{u},\vec{v}}} \right\}}{\deg_{\text{in}}(f_{\vec{u},\vec{v}})} > \gamma(b_{\overrightarrow{h_{i,j}}} + \mathcal{E}_{\overrightarrow{h_{i,j}}}) \\ & \equiv \min \left\{ \frac{\min \left\{ \Phi(\deg(u) - 1 + \mathcal{E}_{\vec{u}}) - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u) \right\}}{\deg(u)} + \frac{\min \left\{ \Phi(\deg(v) - 1 + \mathcal{E}_{\vec{v}}) - \gamma(\deg(v) + \mathcal{E}_{\vec{v}}), \deg(v) \right\}}{\deg(v)} - \gamma(1 + \mathcal{E}_{f_{\vec{u},\vec{v}}}), 1 \right\} > \gamma \mathcal{E}_{\overrightarrow{h_{i,j}}} \end{aligned}$$

These constraints are satisfied provided the inequalities (Equation 2.18)–(Equation 2.21) are satisfied, and the following holds:

$$\boxed{\frac{\Phi(\deg(u) - 1 + \mathcal{E}_{\vec{u}}) - \gamma(\deg(u) + \mathcal{E}_{\vec{u}})}{\deg(u)} + \frac{\Phi(\deg(v) - 1 + \mathcal{E}_{\vec{v}}) - \gamma(\deg(v) + \mathcal{E}_{\vec{v}})}{\deg(v)} > \gamma(1 + \mathcal{E}_{f_{\vec{u},\vec{v}}})} \quad (2.22)$$

$$\Phi(\deg(u) - 1 + \mathcal{E}_u^{\rightarrow}) - \gamma(\deg(u) + \mathcal{E}_u^{\rightarrow}) \leq \deg(u) \equiv \boxed{\Phi \leq \gamma \left(\frac{\deg(u) + \mathcal{E}_u^{\rightarrow}}{\deg(u) - 1 + \mathcal{E}_u^{\rightarrow}} \right) + \frac{\deg(u)}{\deg(u) - 1 + \mathcal{E}_u^{\rightarrow}}} \quad (2.23)$$

$$\boxed{\frac{\Phi(\deg(u) - 1 + \mathcal{E}_u^{\rightarrow}) - \gamma(\deg(u) + \mathcal{E}_u^{\rightarrow})}{\deg(u)} + \frac{\Phi(\deg(v) - 1 + \mathcal{E}_v^{\rightarrow}) - \gamma(\deg(v) + \mathcal{E}_v^{\rightarrow})}{\deg(v)} - \gamma(1 + \mathcal{E}_{f_{u,v}}^{\rightarrow}) > \gamma \mathcal{E}_{h_{i,j}}^{\rightarrow}} \quad (2.24)$$

$$\boxed{\gamma \mathcal{E}_{h_{i,j}}^{\rightarrow} < 1} \quad (2.25)$$

(III) When the node v_{side} is shocked at $t = 1$, the following happens.

(III-a) v_{side} fails at $t = 1$:

$$\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) > \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) \equiv \Phi(|F| + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) > \gamma(|F| + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) \equiv \Phi > \gamma$$

which is same as (Equation 2.18).

(III-b) If a node $\varpi_j \in \varpi$ is shocked at $t = 1$, it does not fail:

$$\Phi(b_{\varpi_j} - \iota_{\varpi_j} + \mathcal{E}_{\varpi_j}^{\rightarrow}) \leq \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}^{\rightarrow}) \equiv \boxed{\Phi \leq \gamma \left(1 + \frac{1}{\mathcal{E}_{\varpi_j}^{\rightarrow}} \right)} \quad (2.26)$$

(III-c) Any node $\varpi_j \in \varpi$ fails at $t = 2$ irrespective of whether ϖ_j was shocked or not:

$$\frac{\min\{\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}), b_{v_{\text{side}}}\}}{\deg_{\text{in}}(v_{\text{side}})} > \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}^{\rightarrow})$$

These constraints are satisfied provided:

$$\frac{\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow})}{\deg_{\text{in}}(v_{\text{side}})} > \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}^{\rightarrow}) \equiv \boxed{\Phi > \frac{\gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}^{\rightarrow}}{|F|} + \mathcal{E}_{\varpi_j}^{\rightarrow} \right)}{\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}^{\rightarrow}}{|F|} \right)}} \quad (2.27)$$

$$\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}^{\rightarrow}) \leq b_{v_{\text{side}}} \equiv \boxed{\Phi \leq \gamma + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}^{\rightarrow}}{|F|}}} \quad (2.28)$$

(III-d) If a node $\varpi_j \in \varpi$ is shocked at $t = 1$, it does not fail (and thus, by **(III-b)**, it does not fail at $t = 2$ also):

$$\Phi(b_{\varpi_j} - \iota_{\varpi_j} + \mathcal{E}_{\varpi_j}) \leq \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}) \equiv \boxed{\Phi \leq \gamma \left(1 + \frac{1}{\mathcal{E}_{\varpi_j}}\right)} \quad (2.29)$$

(III-e) Any node $\varpi_j \in \varpi$ fails at $t = 3$ irrespective of whether ϖ_j was shocked or not:

$$\begin{aligned} & \min \left\{ \frac{\min\{\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}), b_{v_{\text{side}}}\}}{\deg_{\text{in}}(v_{\text{side}})} - \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}), b_{\varpi_j} \right\} \\ & \frac{\quad}{\deg_{\text{in}}(\varpi_j)} > \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}) \equiv \\ & \min \left\{ \frac{\min\{\Phi(|F| + \mathcal{E}_{v_{\text{side}}}) - \gamma(|F| + \mathcal{E}_{v_{\text{side}}}), |F|\}}{|F|} - \gamma(1 + \mathcal{E}_{\varpi_j}), 1 \right\} > \gamma(1 + \mathcal{E}_{\varpi_j}) \end{aligned}$$

These constraints are satisfied provided all the previous constraints hold and the following holds:

$$\frac{\Phi(|F| + \mathcal{E}_{v_{\text{side}}}) - \gamma(|F| + \mathcal{E}_{v_{\text{side}}})}{|F|} - \gamma(1 + \mathcal{E}_{\varpi_j}) > \gamma(1 + \mathcal{E}_{\varpi_j}) \equiv \boxed{\Phi > \gamma \left(\frac{(|F| + \mathcal{E}_{v_{\text{side}}})}{3|F| + |F|\mathcal{E}_{\varpi_j} + |F|\mathcal{E}_{\varpi_j} + \mathcal{E}_{v_{\text{side}}}} \right)} \quad (2.30)$$

$$\frac{\Phi(|F| + \mathcal{E}_{v_{\text{side}}}) - \gamma(|F| + \mathcal{E}_{v_{\text{side}}})}{|F|} - \gamma(1 + \mathcal{E}_{\varpi_j}) \leq 1 \equiv \boxed{\Phi \leq \gamma \left(1 + \frac{1 + \mathcal{E}_{\varpi_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}} \right) + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}}} \quad (2.31)$$

(III-f) Consider a directed path $v_{\text{side}} \leftarrow \varpi_j \leftarrow \varpi_j \leftarrow p_j$ from $p_j = f_{\vec{u}, \vec{v}}$ to v_{side} . The maximum value of its proportion of shock receive by p_j from this path, say σ_2 , is obtained by shocking all the nodes $v_{\text{side}}, \varpi_j, \varpi_j$ and is given by (assuming all previous inequalities hold):

$$\begin{aligned} \sigma_2 &= \min \left\{ \frac{\min \left\{ \frac{\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}})}{\deg_{\text{in}}(v_{\text{side}})} - (\gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}) - \Phi(b_{\varpi_j} - \iota_{\varpi_j} + \mathcal{E}_{\varpi_j})), b_{\varpi_j} \right\}}{\deg_{\text{in}}(\varpi_j)} - (\gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}) - \Phi(b_{\varpi_j} - \iota_{\varpi_j} + \mathcal{E}_{\varpi_j})), b_{\varpi_j} \right\} \\ &= \min \left\{ \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right), 1 \right\} - (\gamma(1 + \mathcal{E}_{\varpi_j}) - \Phi \mathcal{E}_{\varpi_j}), 1 \right\} \end{aligned}$$

Similarly, the minimum value of its proportion of shock receive by p_j from this path, say σ_2 , is obtained by shocking only the node v_{side} and is given by (assuming all previous inequalities hold):

$$\begin{aligned} \sigma'_2 &= \frac{\min \left\{ \frac{\Phi(b_{v_{\text{side}}} - \iota_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}}) - \gamma(b_{v_{\text{side}}} + \mathcal{E}_{v_{\text{side}}})}{\text{deg}_{\text{in}}(v_{\text{side}})} - \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}), \frac{\Phi(b_{\varpi_j} + \mathcal{E}_{\varpi_j})}{\text{deg}_{\text{in}}(\varpi_j)} - \gamma(b_{\varpi_j} + \mathcal{E}_{\varpi_j}), b_{\varpi_j} \right\}}{\text{deg}_{\text{in}}(\varpi_j)} \\ &= \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - \gamma(1 + \mathcal{E}_{\varpi_j}), 1 \right\} \end{aligned}$$

We want node $f_{\vec{u}, \vec{v}}$ to fail at $t = 4$ assuming it did not fail already. Since $f_{\vec{u}, \vec{v}}$ did not fail at $t = 2$, at most one of the nodes \vec{u} and \vec{v} was shocked. There are two cases to consider: when neither \vec{u} nor \vec{v} was shocked, or when exactly one of these nodes, say \vec{v} , was shocked (assuming all previous inequalities hold):

$$\begin{aligned} \sigma'_2 + \sigma_1 &= \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - \gamma(1 + \mathcal{E}_{\varpi_j}), 1 \right\} \\ &\quad + \frac{\min \left\{ \frac{\min \left\{ \Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n \right\}}{n} - \gamma(\text{deg}(u) + \mathcal{E}_{\vec{u}}), \text{deg}(u) \right\}}{\text{deg}(u)} \\ &\quad + \frac{\min \left\{ \frac{\min \left\{ \Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n \right\}}{n} - \gamma(\text{deg}(v) + \mathcal{E}_{\vec{v}}), \text{deg}(v) \right\}}{\text{deg}(v)} \\ &> \gamma(b_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \\ &\equiv \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - \gamma(1 + \mathcal{E}_{\varpi_j}), 1 \right\} \\ &\quad + \frac{\min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma(\text{deg}(u) + \mathcal{E}_{\vec{u}}), \text{deg}(u) \right\}}{\text{deg}(u)} \\ &\quad + \frac{\min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma(\text{deg}(v) + \mathcal{E}_{\vec{v}}), \text{deg}(v) \right\}}{\text{deg}(v)} \\ &> \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \end{aligned}$$

$$\begin{aligned}
& \sigma'_2 + \sigma'_1 + \frac{\min\left\{\Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}}\right\}}{\deg_{\text{in}}(\vec{v})} > \gamma(b_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \\
& \equiv \min\left\{\Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}\right) - \gamma\left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j}\right) - \gamma(1 + \mathcal{E}_{\varpi_j}), 1\right\} \\
& \quad + \frac{\min\left\{\frac{\min\left\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\right\}}{n} - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u)\right\}}{\deg(u)} \\
& \quad + \frac{\min\left\{\Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}}\right\}}{\deg_{\text{in}}(\vec{v})} \\
& > \gamma(b_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \\
& \equiv \min\left\{\Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}\right) - \gamma\left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j}\right) - \gamma(1 + \mathcal{E}_{\varpi_j}), 1\right\} \\
& \quad + \frac{\min\left\{\Phi\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u)\right\}}{\deg(u)} \\
& \quad + \frac{\min\left\{\Phi(\deg(\vec{v}) - 1 + \mathcal{E}_{\vec{v}}) - \gamma(\deg(\vec{v}) + \mathcal{E}_{\vec{v}}), \deg(\vec{v})\right\}}{\deg_{\text{in}}(\vec{v})} \\
& > \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}})
\end{aligned}$$

These constraints are satisfied provided all the previous constraints hold and the following holds:

$$\begin{aligned}
& \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}\right) - \gamma\left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j}\right) - \gamma(1 + \mathcal{E}_{\varpi_j}) \\
& \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}) \\
& + \frac{\deg(u)}{\deg(u)} \\
& \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma\left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}\right) - \gamma(\deg(v) + \mathcal{E}_{\vec{v}}) \\
& + \frac{\deg(v)}{\deg(v)} > \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}})
\end{aligned}$$

$$\equiv \Phi > \gamma \left(\frac{6 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \mathcal{E}_{f_{\vec{u}, \vec{v}}} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varrho_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)}} \right) \quad (2.32)$$

$$\Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - \gamma \left(1 + \mathcal{E}_{\varrho_j} \right) \leq 1 \equiv \Phi \leq \gamma \left(\frac{3 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varrho_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}} \right) + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|}} \quad (2.33)$$

$$\Phi \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(\deg(u) + \mathcal{E}_{\vec{u}} \right) \leq \deg(u) \equiv \Phi \leq \gamma \left(\frac{1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} + \deg(u) + \mathcal{E}_{\vec{u}}}{1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}} \right) + \frac{\deg(u)}{1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n}} \quad (2.34)$$

$$\Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - \gamma \left(1 + \mathcal{E}_{\varrho_j} \right) + \frac{\Phi \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(1 + \frac{\mathcal{E}_{v_{\text{top}}}}{n} \right) - \gamma \left(\deg(u) + \mathcal{E}_{\vec{u}} \right)}{\deg(u)} + \frac{\Phi \left(\deg(\vec{v}) - 1 + \mathcal{E}_{\vec{v}} \right) - \gamma \left(\deg(\vec{v}) + \mathcal{E}_{\vec{v}} \right)}{\deg_{\text{in}}(\vec{v})} > \gamma \left(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}} \right) \quad (2.35)$$

$$\equiv \Phi > \gamma \left(\frac{6 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varrho_j} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}} + 1}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}}{2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{v}} - 1}{\deg(v)}} \right) \quad (2.36)$$

(IV) By (II-b) node $\vec{h}_{i,j}$ fails at $t = 3$ provided both the nodes \vec{u} and \vec{v} were shocked at $t = 1$.

The goal is to make sure that node $\vec{h}_{i,j}$ does not fail in any other condition (assuming the node itself was not shocked). Assuming the nodes \vec{u} , \vec{v} and $f_{\vec{u}, \vec{v}}$ were not shocked, the maximum amount of shock that $f_{\vec{u}, \vec{v}} \in \vec{F}_{i,j}$ can receive is when all the nodes before $f_{\vec{u}, \vec{v}}$ in the path

$v_{\text{side}} \leftarrow \varpi_j \leftarrow \varpi_j \leftarrow p_j \leftarrow f_{\vec{u}, \vec{v}}$ were shocked and no more than one of the nodes \vec{u} or \vec{v} was shocked.

Based on this, the following constraints must hold for $\vec{h}_{i,j}$ not to fail.

$$\begin{aligned}
& \frac{\min\{\sigma_1 + \sigma_2 - \gamma(b_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}), b_{f_{\vec{u}, \vec{v}}}\}}{\deg_{\text{in}}(f_{\vec{u}, \vec{v}})} \leq \frac{\gamma(b_{\vec{h}_{i,j}} + \mathcal{E}_{\vec{h}_{i,j}})}{|F_{i,j}|} \\
\equiv & \min \left\{ \frac{\min\left\{ \frac{\min\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\}}{n} - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u) \right\}}{\deg(u)} \right. \\
& \left. \frac{\min\left\{ \frac{\min\{\Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n\}}{n} - \gamma(\deg(v) + \mathcal{E}_{\vec{v}}), \deg(v) \right\}}{\deg(v)} \right. \\
& \left. + \min\left\{ \min\left\{ \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j}\right) - \gamma\left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j}\right), 1\right\} - (\gamma(1 + \mathcal{E}_{\varpi_j}) - \Phi\mathcal{E}_{\varpi_j}), 1\right\} \right. \\
& \left. - \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}}), 1\right\} \leq \frac{\gamma \mathcal{E}_{\vec{h}_{i,j}}}{|F_{i,j}|} \\
\equiv & \min \left\{ \Phi\left(\frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)}\right) \right. \\
& \left. - \gamma\left(3 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \mathcal{E}_{f_{\vec{u}, \vec{v}}}\right) \right. \\
& \left. + \min\left\{ \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}\right) - \gamma\left(3 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}\right), 1\right\}, 1\right\} \leq \frac{\gamma \mathcal{E}_{\vec{h}_{i,j}}}{|F_{i,j}|}
\end{aligned}$$

These constraints are satisfied provided all the previous constraints hold and the following holds:

$$\begin{aligned}
& \Phi\left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}\right) - \gamma\left(3 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}\right) \leq 1 \\
\equiv & \boxed{\Phi \leq \gamma \left(\frac{3 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}} \right) + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\varpi_j}}} \quad (2.37)
\end{aligned}$$

$$\Phi \leq \gamma \left(\frac{6 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \mathcal{E}_{f_{\vec{u},\vec{v}}} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}}{1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) + \frac{1}{1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \quad (2.38)$$

$$\Phi \leq \gamma \left(\frac{6 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \mathcal{E}_{f_{\vec{u},\vec{v}}} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j} + \frac{\mathcal{E}_{h_{i,j}}}{|F_{i,j}|}}{1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{1}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) \quad (2.39)$$

On the other hand, if exactly one of the nodes \vec{u} or \vec{v} , say \vec{u} , was shocked at $t = 1$, then the maximum amount of shock that $f_{\vec{u},\vec{v}} \in \vec{F}_{i,j}$ can receive is modified, and the new conditions for the desired goal become as follows.

$$\begin{aligned} & \min \left\{ \sigma'_1 + \frac{\min \left\{ \Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}} \right\}}{\deg_{\text{in}}(\vec{v})} + \sigma_2 - \gamma(b_{f_{\vec{u},\vec{v}}} + \mathcal{E}_{f_{\vec{u},\vec{v}}}), b_{f_{\vec{u},\vec{v}}} \right\} \leq \frac{\gamma(b_{h_{i,j}} + \mathcal{E}_{h_{i,j}})}{|F_{i,j}|} \\ & \equiv \min \left\{ \frac{\min \left\{ \frac{\min \left\{ \Phi(n + \mathcal{E}_{v_{\text{top}}}) - \gamma(n + \mathcal{E}_{v_{\text{top}}}), n \right\}}{n} - \gamma(\deg(u) + \mathcal{E}_{\vec{u}}), \deg(u) \right\}}{\deg(u)} \right. \\ & \quad + \frac{\min \left\{ \Phi(b_{\vec{v}} - \iota_{\vec{v}} + \mathcal{E}_{\vec{v}}) - \gamma(b_{\vec{v}} + \mathcal{E}_{\vec{v}}), b_{\vec{v}} \right\}}{\deg_{\text{in}}(\vec{v})} \\ & \quad \left. + \min \left\{ \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right), 1 \right\} - (\gamma(1 + \mathcal{E}_{\mathfrak{m}_j}) - \Phi \mathcal{E}_{\mathfrak{m}_j}), 1 \right\} \right. \\ & \quad \left. - \gamma(1 + \mathcal{E}_{f_{\vec{u},\vec{v}}}), 1 \right\} \leq \frac{\gamma \mathcal{E}_{h_{i,j}}}{|F_{i,j}|} \end{aligned}$$

$$\begin{aligned} \equiv \min & \left\{ \Phi \left(1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) - \gamma \left(2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) \right. \\ & \left. + \min \left\{ \min \left\{ \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right), 1 \right\} - (\gamma(1 + \mathcal{E}_{\mathfrak{m}_j}) - \Phi \mathcal{E}_{\mathfrak{m}_j}), 1 \right\} \right. \\ & \left. - \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}}), 1 \right\} \leq \frac{\gamma \mathcal{E}_{h_{i,j}}}{|F_{i,j}|} \end{aligned}$$

These constraints are satisfied provided all the previous constraints hold and the following holds:

$$\Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) \leq 1 \equiv \Phi \leq \gamma \left(\frac{2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j}} \right) + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j}} \quad (2.40)$$

$$\begin{aligned} & \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - (\gamma(1 + \mathcal{E}_{\mathfrak{m}_j}) - \Phi \mathcal{E}_{\mathfrak{m}_j}) \leq 1 \\ & \equiv \Phi \leq \gamma \left(\frac{2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + 1 + \mathcal{E}_{\mathfrak{m}_j}}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) + \frac{1}{1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \quad (2.41) \end{aligned}$$

$$\begin{aligned} & \Phi \left(1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) - \gamma \left(2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) \\ & + \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - (\gamma(1 + \mathcal{E}_{\mathfrak{m}_j}) - \Phi \mathcal{E}_{\mathfrak{m}_j}) - \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \leq \frac{\gamma \mathcal{E}_{h_{i,j}}}{|F_{i,j}|} \\ & \equiv \Phi \leq \gamma \left(\frac{6 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{\mathfrak{m}_j} + \frac{\gamma \mathcal{E}_{h_{i,j}}}{|F_{i,j}|}}{2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) \quad (2.42) \end{aligned}$$

$$\begin{aligned} & \Phi \left(1 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) - \gamma \left(2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} \right) \\ & + \Phi \left(1 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} \right) - \gamma \left(2 + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} \right) - (\gamma(1 + \mathcal{E}_{\mathfrak{m}_j}) - \Phi \mathcal{E}_{\mathfrak{m}_j}) - \gamma(1 + \mathcal{E}_{f_{\vec{u}, \vec{v}}}) \leq 1 \quad (2.43) \end{aligned}$$

$$\begin{aligned}
& \equiv \left(\Phi \leq \gamma \left(\frac{6 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} + \frac{\mathcal{E}_{\vec{u}}}{\deg(u)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} + \mathcal{E}_{\varpi_j} + \mathcal{E}_{f_{\vec{u}, \vec{v}}} + \mathcal{E}_{\mathfrak{m}_j}}{2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) \right. \\
& \quad \left. + \frac{1}{2 + \frac{1}{\deg(u)} + \frac{\mathcal{E}_{v_{\text{top}}}}{n \deg(u)} - \frac{1}{\deg(v)} + \frac{\mathcal{E}_{\vec{v}}}{\deg(v)} + \frac{\mathcal{E}_{v_{\text{side}}}}{|F|} - \mathcal{E}_{\varpi_j} + \mathcal{E}_{\mathfrak{m}_j}} \right) \quad (2.44)
\end{aligned}$$

There are many choices of parameters γ , Φ and \mathcal{E}_y 's satisfying inequalities (Equation 2.18)–(Equation 2.27); just one is exhibited:

$$\begin{aligned}
\gamma &= 4n^{-1000} & \Phi &= n^{-1000} & \forall u \in V^{\text{left}} \cup V^{\text{right}}: \mathcal{E}_{\vec{u}} &= 1 & \mathcal{E}_{v_{\text{top}}} &= n^3 & \mathcal{E}_{v_{\text{side}}} &= n^2 |F| \\
\forall u \in V^{\text{left}} \forall v \in V^{\text{right}}: \mathcal{E}_{f_{\vec{u}, \vec{v}}} &= 1 & \forall h_{i,j} \in F^{\text{super}} \forall f_{u,v} \in F_{i,j}: \mathcal{E}_{h_{i,j}} &= 1 & \forall j: \mathcal{E}_{\varpi_j} &= \mathcal{E}_{\mathfrak{m}_j} &= \frac{1}{4}
\end{aligned}$$

Remembering that $10 \leq \deg(u) < n$ for any node $u \in V^{\text{left}} \cup V^{\text{right}}$ and $|F_{i,j}| < |F|$, it is relatively straightforward to verify that all the inequalities are satisfied for all sufficiently large n . Note that

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_{v_{\text{top}}} + \mathcal{E}_{v_{\text{side}}} + \sum_{u \in V^{\text{left}} \cup V^{\text{right}}} \mathcal{E}_{\vec{u}} + \sum_{\{u,v\} \in F} \mathcal{E}_{f_{\vec{u}, \vec{v}}} + \sum_{h_{i,j} \in F^{\text{super}}} \mathcal{E}_{h_{i,j}} + \sum_{j=1}^{|F|} (\mathcal{E}_{\varpi_j} + \sum \mathcal{E}_{\mathfrak{m}_j}) \\
&= n^3 + n^2 |F| + n + \frac{3}{2} |F| + |F^{\text{super}}|
\end{aligned}$$

and thus the ratio of total external assets to total internal assets \mathcal{E}/\mathcal{I} is large. The proof can be finished by selecting δ such that $\log^{1-\delta} n = \log^{1-\varepsilon} |\vec{V}| - 1$ and showing the following:

(completeness) If MINREP has a solution of size $\alpha + \beta$ on G then $\text{Sl}^*(\vec{G}, T) \leq \alpha + \beta + 2$.

(soundness) If every solution of MINREP on G is of size at least $(\alpha + \beta)2^{\log^{1-\delta} n}$ then

$$\text{Sl}^*(\vec{G}, T) \geq \frac{\alpha + \beta}{2} 2^{\log^{1-\delta} n}.$$

Proof of Completeness (MINREP has a solution of size $\alpha + \beta$)

Let $V_1 \subseteq V^{\text{left}}$ and $V_2 \subseteq V^{\text{right}}$ be a solution of MINREP such that $|V_1| + |V_2| = \alpha + \beta$. The nodes v_{top} and v_{side} are shocked, and every node \vec{u} for every $u \in V^{\text{left}} \cup V^{\text{right}}$ are shocked. By **(I-a)** v_{top} fails at $t = 1$, and by **(I-b)** and **(II-a)** every node in $\cup_{i=1}^{\alpha} V_i^{\text{left}} \cup \cup_{j=1}^{\beta} V_j^{\text{right}}$ fails on or before $t = 2$. By **(III-a)**, **(III-b)** and **(III-c)** every node in $\{V_{\text{shock}}\} \cup \mathfrak{B} \cup \mathfrak{M}$ fails on or before $t = 3$. Since V_1 and V_2 are a valid solution of MINREP, for every super-edge $h_{i,j}$ there exists $u \in V_1$ and $v \in V_2$ such that $u \in V_i^{\text{left}}$, $v \in V_j^{\text{right}}$ and $\{u, v\} \in F$; since the nodes \vec{u} and \vec{v} are shocked, by **(II-a)** both \vec{u} and \vec{v} fail at $t = 1$, by **(II-b)** the node $f_{\vec{u}, \vec{v}}$ fails at $t = 2$, and by **(II-c)** the node $\vec{h}_{i,j}$ fails at $t = 3$. Thus, the network \vec{G} fails at $t = 3$ and $\text{SI}^*(\vec{G}, T) = \alpha + \beta + 1$ for $t \geq 4$.

Proof of Soundness (every solution of MINREP is of size at least $(\alpha + \beta)2^{\log^{1-\delta} n}$)

The logically equivalent contrapositive of the claim will be proved, *i.e.*, if $\text{NVI}^*(\vec{G}, T) < \frac{\alpha + \beta}{2} 2^{\log^{1-\delta} n}$ then MINREP has a solution of size strictly less than $(\alpha + \beta)2^{\log^{1-\delta} n}$. Consider a solution of $\text{STAB}_{T, \phi}$ on \vec{G} that shocks at most $z = \frac{\alpha + \beta}{2} 2^{\log^{1-\delta} n}$ nodes. Note that the nodes v_{top} and v_{side} must be shocked at $t = 1$ by Proposition 2.8.1(a). By **(I-a)** and **(III-a)**, the nodes v_{top} and v_{side} fails at $t = 1$, by **(I-b)** and **(III-c)** every node in $\vec{V}^{\text{left}} \cup \vec{V}^{\text{right}} \cup \mathfrak{B}$ fails at $t = 2$, by **(III-e)** every node in \mathfrak{M} fails at $t = 3$, by **(III-f)** every node $f_{\vec{u}, \vec{v}}$ fails at $t = 4$ unless it was shocked at $t = 1$ and by **(IV)** a node $\vec{h}_{i,j}$ fails only if $\vec{h}_{i,j}, f_{\text{ovru}, \text{ovru}} \in \vec{F}_{i,j}$ or *both* the nodes \vec{u} and \vec{v} were shocked at $t = 1$. This given solution is “normalize’d” in the following manner (each step of the normalization assumes that the previous steps have been already carried out):

- If a node from $\mathfrak{M} \cup \mathfrak{B}$ was shocked at $t = 1$, do not shock it. By **(III)** this has no effect on the failure of the network.
- If a node $f_{\vec{u}, \vec{v}} \in \vec{F}_{i,j}$ was shocked, do not shock it but instead shock the nodes \vec{u} and \vec{v} if they were not already shocked in the given solution. This at most doubles the number of nodes shocked and, by **(II-b)**, the node $f_{u,v}$ fails at $t = 2$ and the node $\vec{h}_{i,j}$ fails at $t = 3$ if it was not shocked at $t = 1$. Thus, after this sequence of normalization steps, assume that no $f_{\vec{u}, \vec{v}}$ node was shocked.

- If a node $\vec{h}_{i,j}$ was shocked at $t = 1$, do not shock it but instead shock the nodes \vec{u} and \vec{v} (for some u and v such that $\{u, v\} \in F_{i,j}$) if they were not already shocked in the given solution. This at most doubles the number of nodes shocked and, by **(II-b)**, the node $f_{u,v}$ fails at $t = 2$ and the node $\vec{h}_{i,j}$ fails at $t = 3$. Thus, after this sequence of normalization steps, assume that no $\vec{h}_{i,j}$ node was shocked.

These normalizations result in a solution of $\text{STAB}_{T,\phi}$ of size at most $2z$ in which the nodes $v_{\text{top}}, v_{\text{side}}$, a subset $\vec{V}_1 \subseteq \vec{V}^{\text{left}}$ and a subset $\vec{V}_2 \subseteq \vec{V}^{\text{right}}$ of nodes. The solution of MINREP is $V_1 = \{v \mid \vec{v} \in \vec{V}_1\} \subseteq V^{\text{left}}$ and $V_2 = \{v \mid \vec{v} \in \vec{V}_2\} \subseteq V^{\text{right}}$ of size $2z - 2 < 2z$. Since failure of every $\vec{h}_{i,j}$ is attributed to shocking two nodes \vec{u} and \vec{v} such that $f_{\vec{u},\vec{v}} \in \vec{F}_{i,j}$, every super-edge $h_{i,j}$ of G is witnessed by the two nodes u and v . \square

2.8.8 Homogeneous Networks, $\text{DUAL-STAB}_{T,\phi,\kappa}$, any T , hardness and exact algorithm

Theorem 2.8.14.

(a) Assuming $\text{P} \neq \text{NP}$, $\text{DSI}^*(G, T, \kappa)$ cannot be approximated within a factor of $(1 - e^{-1} + \delta)^{-1}$, for any $\delta > 0$, even if G is a DAG (e is the base of natural logarithm).

(b) If G is a rooted in-arborescence then $\text{DSI}^*(G, T, \kappa) < \frac{\kappa}{n} \left(1 + \deg_{\text{in}}^{\max} \left(\frac{\Phi}{\gamma} - 1\right)\right)$, where $\deg_{\text{in}}^{\max} = \max_{v \in V} \{\deg_{\text{in}}(v)\}$ is the maximum in-degree over all nodes of G . Moreover, under the assumption that any individual node of the network can be failed by shocking, $\text{DSI}^*(G, T, \kappa)$ can be computed exactly in $O(n^3)$ time.

Proof.

(a) The max κ -cover problem is defined as follows. An instance of the problem is an universe \mathcal{U} of n elements, a collection of m sets \mathcal{S} over \mathcal{U} , and a positive integer κ . The goal is to pick a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ of κ sets such that the number of elements covered, namely $|\cup_{S \in \mathcal{S}'} S|$, is maximized. Let OPT denote the maximum number of elements covered by an optimal solution of the max κ -cover problem. It was shown in (45) that, assuming $\text{P} \neq \text{NP}$, the max κ -cover problem cannot be approximated within a factor of $\frac{1}{(1 - \frac{1}{e} + \delta)}$ for any constant $\delta > 0$. More precisely, (45) provides a polynomial-time reduction for a

restricted but still NP-hard version of the Boolean satisfiability problem (3-CNF5) instances of max κ -cover with $\kappa = |\mathcal{U}|^\alpha$, for some constant $0 < \alpha < 1$, and shows that

- (1) if the CNF formula is satisfiable, then $\text{OPT} = |\mathcal{U}|$;
- (2) if the CNF formula is not satisfiable, then $\text{OPT} < \left(1 - \frac{1}{e} + g(\kappa)\right) |\mathcal{U}|$, where $g(\kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$.

The reduction from max κ -cover to $\text{Dual-Stab}_{T,\kappa}$ is as follows¹. In the graph $G = (V, F)$, we have an element node \tilde{u} for every element $u \in \mathcal{U}$, a set node \tilde{S} for every set $S \in \mathcal{S}$, and directed edges (\tilde{u}, \tilde{S}) for every element $u \in \mathcal{U}$ and set $S \in \mathcal{S}$ such that $u \in S$. Thus, $n = |V| = |\mathcal{U}| + |\mathcal{S}|$ and $|F| = \sum_{S \in \mathcal{S}} |S|$. The remaining parameters are set as follows: $\mathcal{E} = n$, $\gamma = n^{-2}$ and $\Phi = 1$. Now, the following can be observed:

- If an element node \tilde{u} is shocked, it does not fail since $\Phi \left(\text{deg}_{\text{in}}(\tilde{u}) - \text{deg}_{\text{out}}(\tilde{u}) + \frac{\mathcal{E}}{n} \right) \leq 0$ whereas $\gamma \left(\text{deg}_{\text{in}}(\tilde{u}) + \frac{\mathcal{E}}{n} \right) = n^{-2} > 0$.
- If a set node \tilde{S} is shocked, it fails since $\Phi \left(\text{deg}_{\text{in}}(\tilde{S}) - \text{deg}_{\text{out}}(\tilde{S}) + \frac{\mathcal{E}}{n} \right) \geq 2$ whereas $\gamma \left(\text{deg}_{\text{in}}(\tilde{S}) + \frac{\mathcal{E}}{n} \right) \leq \frac{n+1}{n^2} < 1$.
- If a set node \tilde{S} is shocked, then every element node \tilde{u} for $u \in S$ fails at $t = 2$. To observe this, note that

$$\begin{aligned} & \frac{\min \left\{ \Phi \left(\text{deg}_{\text{in}}(\tilde{S}) - \text{deg}_{\text{out}}(\tilde{S}) + \frac{\mathcal{E}}{n} \right) - \gamma \left(\text{deg}_{\text{in}}(\tilde{u}) + \frac{\mathcal{E}}{n} \right), \text{deg}_{\text{in}}(\tilde{S}) \right\}}{\text{deg}_{\text{in}}(\tilde{S})} \\ & \geq \frac{2 - \frac{n+1}{n^2}}{n} > \frac{n+1}{n^2} \geq \gamma \left(\text{deg}_{\text{in}}(\tilde{S}) + \frac{\mathcal{E}}{n} \right) \end{aligned}$$

- Since the longest directed path in G has one edge, no new nodes fails during $t > 2$.

¹However, this exact construction will not work in the proof of Theorem 2.8.2 since the entire network needs to fail in that proof.

Based on the above observations, one can identify the sets selected in max k -cover with the set nodes selected for shocking in $\text{Dual-Stab}_{T,\kappa}$ on G to conclude that $\text{DSI}^*(G, T, \kappa) = \text{OPT} + \kappa$. Thus, using (1) and (2), inapproximability gap is

$$\frac{|\mathcal{U}| + \kappa}{\left(1 - \frac{1}{e} + g(\kappa)\right)|\mathcal{U}| + \kappa} = \frac{|\mathcal{U}| + |\mathcal{U}|^\alpha}{\left(1 - \frac{1}{e} + g(\kappa)\right)|\mathcal{U}| + |\mathcal{U}|^\alpha} \rightarrow \frac{1}{1 - \frac{1}{e} + \delta} \text{ as } |\mathcal{U}| \rightarrow \infty \text{ for any } \delta > 0$$

(b) Using Lemma 2.8.8, we have

$$\text{DSI}^*(G, T, \kappa) < \frac{\kappa \left(\max_{u \in V} \{ \text{iz}(u) \} \right)}{n} < \frac{\kappa}{n} \left(1 + \text{deg}_{\text{in}}^{\max} \left(\frac{\Phi}{\gamma} - 1 \right) \right)$$

To provide a polynomial time algorithm for $\text{DSI}^*(G, T, \kappa)$, the algorithm described in the proof of Theorem 2.8.5 is modified suitably. $\text{SI}_{\text{SANS}}^*(G, T, u, \nu)$ and $\text{SI}_{\text{SAS}}^*(G, T, u)$ are redefined in the following manner:

- For every node $u' \in \nabla(u)$ and every integer $0 \leq k \leq \kappa$, $\text{DSI}_{\text{SANS}}^*(G, T, u, u', k)$ is the number of nodes in an optimal solution of $\text{DUAL-STAB}_{T,\phi,\kappa}$ (or ∞ if there is no feasible solution of $\text{DUAL-STAB}_{T,\phi,\kappa}$) for the subgraph induced by the nodes in $\Delta(u)$ assuming the following:
 - u' was shocked,
 - u was not shocked,
 - no node in the path $u' \rightsquigarrow u$ except u' was shocked, and
 - total number of shocked nodes in $\Delta(u)$ is exactly k .
- For every integer $0 \leq k \leq \kappa$, $\text{DSI}_{\text{SAS}}^*(G, T, u, k)$ is the number of nodes in an optimal solution of $\text{DUAL-STAB}_{T,\phi,\kappa}$ for the subgraph induced by the nodes in $\Delta(u)$ (or ∞ , if there is no feasible solution of $\text{STAB}_{T,\phi}$ under the stated conditions) assuming that the node u was shocked (and therefore failed), and the number of shocked nodes in $\Delta(u)$ is exactly k .

Computing these quantities becomes slightly more computationally involved as shown below.

Computing $DSI_{SAS}^*(G, T, u, k)$ when $\deg_{in}(u) = 0$:

$$DSI_{SAS}^*(G, T, u, 1) = 1 \text{ and } DSI_{SAS}^*(G, T, u, k) = -\infty \text{ for any } k \neq 1.$$

Computing $DSI_{SANS}^*(G, T, u, u', k)$ when $\deg_{in}(u) = 0$:

- If $u \in iz(u')$ then shocking node v makes node u fail. Thus, $SI_{SANS}^*(G, T, u, u', 1) = 1$ and $SI_{SANS}^*(G, T, u, u', k) = -\infty$ for any $k \neq 1$.
- Otherwise, node u does not fail. Thus, $DSI_{SANS}^*(G, T, u, u') = -\infty$.

Computing $DSI_{SAS}^*(G, T, u)$ when $\deg_{in}(u) > 0$: In this case we have

$$DSI_{SAS}^*(G, T, u, k) = 1 + \min_{k_1+k_2+\dots+k_{\deg_{in}(u)}=k-1} \left\{ \sum_{i=1}^k \min \left\{ DSI_{SAS}^*(G, T, v_i, k_i), DSI_{SANS}^*(G, T, v_i, u, k_i) \right\} \right\}$$

Computing $DSI_{SANS}^*(G, T, u, u', k)$ when $\deg_{in}(u) > 0$: Since u' is shocked and u is not shocked, the following cases arise:

- If $u \notin iz(u')$ then u does not fail. Then,

$$DSI_{SANS}^*(G, T, u, u', k) = \min_{k_1+k_2+\dots+k_{\deg_{in}(u)}=k} \left\{ \sum_{i=1}^{\deg_{in}(u)} \min \left\{ DSI_{SAS}^*(G, T, v_i, k_i), SI_{SANS}^*(G, T, v_i, u', k_i) \right\} \right\}$$

- Otherwise, $u \in \text{iz}(u')$, and therefore u fails when u' is shocked. Then,

$$\text{DSI}_{\text{SANS}}^*(G, T, u, u', k) = 1 + \min_{k_1+k_2+\dots+k_{\deg_{\text{in}}(u)}=k} \left\{ \sum_{i=1}^{\deg_{\text{in}}(u)} \min \left\{ \text{DSI}_{\text{SAS}}^*(G, T, v_i, k_i), \text{DSI}_{\text{SANS}}^*(G, T, v_i, u', k_i) \right\} \right\}$$

It only remains to show how to compute

$\min_{k_1+k_2+\dots+k_{\deg_{\text{in}}(u)}=F} \left\{ \sum_{i=1}^{\deg_{\text{in}}(u)} \min \left\{ \text{DSI}_{\text{SAS}}^*(G, T, v_i, k_i), \text{DSI}_{\text{SANS}}^*(G, T, v_i, u', k_i) \right\} \right\}$ for $F \in \{k-1, k\}$ in polynomial time. It is easy to cast this problem as an instance of the unbounded integral knapsack problem in the

following manner:

- We have $\deg_{\text{in}}(u)$ objects $O_1, O_2, \dots, O_{\deg_{\text{in}}(u)}$, each of *unlimited* supply and *weight* 1.
- The *cost* of selecting k_i objects of the type O_i is

$$\min \left\{ \text{DSI}_{\text{SAS}}^*(G, T, v_i, k_i), \text{DSI}_{\text{SANS}}^*(G, T, v_i, u', k_i) \right\}$$

- The *goal* is to select a total of *exactly* F objects such that the total cost is *minimum*.

The standard pseudo-polynomial time dynamic programming algorithm for Knapsack can be used to solve the above instance in $O(k \deg_{\text{in}}(u)) = O(n^2)$ time. Thus, the total running time of the algorithm is $O(n^3)$. \square

2.8.9 Heterogeneous Networks, **Dual-Stab**_{2,ϕ,κ}, Stronger Inapproximability

Here it is shown that $\text{DSI}^*(G, 2, \kappa)$ cannot be approximated within a large approximation factor provided a complexity-theoretic assumption is satisfied. To understand this assumption, recall the following definitions from (9).

A random (m, n, d) hyper-graph H is a random hyper-graph of n nodes, m hyper-edges each having having exactly d nodes obtained by choosing each hyper-edge independently and uniformly at random.

Assume that d is a constant, and $m \geq n^c$ for some constant $c > 3$. Let $Q: \{0, 1\}^d \mapsto \{0, 1\}$ denote a d -ary predicate, and let $\mathcal{F}_{Q,m}$ be a distribution over d -local functions from $\{0, 1\}^n$ to $\{0, 1\}^m$ by defining the random d -local function $f_{H,Q}: \{0, 1\}^n \mapsto \{0, 1\}^m$ to be the function whose i^{th} output is computed by applying the predicate Q to the d inputs that are indexed by the i^{th} hyper-edge of H . Finally, the κ densest sub-hypergraph problem (DS_κ) is defined as follows: *given an hyper-graph $G = (V, F)$ with $n = |V|$ and $m = |F|$ such that every hyper-edge contains exactly d nodes and an integer $\kappa > 0$, select a subset $V' \subseteq V$ of exactly κ nodes which maximizes $|\{u_1, u_2, \dots, u_d\} \in F \mid u_1, u_2, \dots, u_d \in V'\}|$.*

The essence of the complexity-theoretic assumption is that if, for a suitable choice of Q , $\mathcal{F}_{Q,m}$ is a collection of one-way functions, then DS_κ is hard to approximate. More precisely, the assumption is:

(★) If $\mathcal{F}_{Q,m}$ is $1/o(1/\sqrt{n} \log n)$ -pseudorandom, then for $\kappa = n^{1-\frac{c-3}{2d}}$ for some constant $c > 3$ there exists instances $G = (V, F)$ of DS_κ with $m \geq n^c$ such that it is not possible to decide in polynomial time if there is a solution of DS_κ with at least $\frac{(1+o(1))m}{n^{\frac{(c-3)}{2}(1-\frac{1}{d})}}$ edges (the “yes” instance), or if every solution of DS_κ has at most $\frac{(1-o(1))m}{n^{\frac{c-3}{2}}}$ edges (the “no” instance).

Theorem 2.8.15. *Under the technical assumption (★), $\text{DSI}^*(G, 2, \kappa)$ cannot be approximated within a ratio of n^δ for some constant $\delta > 0$ even if G is a DAG.*

Proof. Given an instance $G = (V, F)$ of DS_κ as stated in (★), an instance graph is constructed $\vec{G} = (\vec{V}, \vec{F})$ as follows:

- For every node $u \in V$, we have a node $\vec{u} \in \vec{V}$, and for every edge $e = \{u_1, u_2, \dots, u_d\} \in F$, we have a node \vec{e} (also denoted by $\overrightarrow{\{u_1, u_2, \dots, u_d\}}$) in \vec{V} . Thus, the total number of nodes of \vec{G} is $|\vec{V}| = m + n$.
- For every hyper-edge $e = (u_1, u_2, \dots, u_d) \in F$, we have d edges $(e, u_1), (e, u_2), \dots, (e, u_d) \in \vec{F}$. The weight (share of internal asset) of every edge (e, u_i) is set to 2. Thus, $|I| = 2dm$.

Let the share of external assets for a node (bank) $\vec{y} \in \vec{V}$ be denoted by $\mathcal{E}_{\vec{y}}$ (thus, $\sum_{\vec{y} \in \vec{V}} \mathcal{E}_{\vec{y}} = \mathcal{E}$). The remaining network parameters are selected as follows. For each $e \in F$, $\mathcal{E}_{\vec{e}} = 1.99d$, and for each $u \in V$, $\mathcal{E}_{\vec{u}} = 0$. Thus, $\mathcal{E} = 1.99dm$. Finally, set $\Phi = 1$ and $\gamma = 1/2$. The following is proved:

(completeness) If DS_{κ} has a solution with $\alpha \geq \frac{(1+o(1))m}{n^{\frac{c-3}{2}(1-\frac{1}{d})}}$ hyper-edges then then

$$\text{DSI}^*(\vec{G}, 2, \kappa) \geq \kappa + \alpha.$$

(soundness) If every solution of DS_{κ} has at most $\beta = \frac{(1-o(1))m}{n^{\frac{c-3}{2}}}$ hyper-edges then

$$\text{DSI}^*(\vec{G}, 2, \kappa) \leq \kappa + \beta.$$

Note that with $c = 5$ (and, thus $m \geq n^5$), and sufficiently large d and n , we have

$$\frac{\kappa + \alpha}{\kappa + \beta} = \frac{n^{1-\frac{c-3}{2d}} + \frac{(1+o(1))m}{n^{\frac{c-3}{2}(1-\frac{1}{d})}}}{n^{1-\frac{c-3}{2d}} + \frac{(1-o(1))m}{n^{\frac{c-3}{2}}}} = \frac{n^{1-\frac{1}{d}} + \frac{(1+o(1))m}{n^{1-\frac{1}{d}}}}{n^{1-\frac{1}{d}} + \frac{(1-o(1))m}{n}} \geq (1 - o(1))n^{1/d}$$

which proves the theorem with $\delta = 1/d$.

Proof of Completeness (DS_{κ} has a solution with α hyper-edges)

Let $V' \subseteq V$ be a solution of DS_{κ} with at least α hyper-edges. All the nodes in $V_{\text{shock}} = \{\vec{u} \mid u \in V'\}$ are shocked. Every shocked node \vec{u} fails at $t = 1$ since $\Phi(b_{\vec{u}} - t_{\vec{u}} + \mathcal{E}_{\vec{u}}) = 2 \deg_{\text{in}}(\vec{u}) > \deg_{\text{in}}(\vec{u}) = \gamma(b_{\vec{u}} + \mathcal{E}_{\vec{u}})$. Now, consider a hyper-edge $e = (u_1, u_2, \dots, u_d) \in F$ such that $u_1, u_2, \dots, u_d \in V'$. Then, the node \vec{e} fails at $t = 2$ since

$$\sum_{i=1}^d \frac{\min\{\Phi(b_{\vec{u}_i} - t_{\vec{u}_i} + \mathcal{E}_{\vec{u}_i}) - \gamma(b_{\vec{u}_i} + \mathcal{E}_{\vec{u}_i}), b_{\vec{u}_i}\}}{\deg_{\text{in}}(\vec{u}_i)} = d > 0.995d = \gamma(b_{\vec{e}} + \mathcal{E}_{\vec{e}})$$

Proof of Soundness (every solution of DS_{κ} has at most β hyper-edges)

The logically equivalent contrapositive of the claim will be proved, *i.e.*, if $\text{DSI}^*(\vec{G}, 2, \kappa) > \beta + \kappa$ then DS_{κ} has a solution of with strictly more than β hyper-edges. First, note that it can be assumed without loss of generality that, for any hyper-edge $e \in F$, the node \vec{e} is not shocked. Otherwise, if node \vec{e} is shocked,

then it does not fail since at $t = 1$ since $\Phi(b_{\vec{e}} - \iota_{\vec{e}} + \mathcal{E}_{\vec{e}}) = -0.01d < 0.995d = \gamma(b_{\vec{e}} + \mathcal{E}_{\vec{e}})$, and in fact doing so increases its equity to $1.005d$. Since the equity of \vec{e} increased by shocking it, if this node failed in the given solution then it would also fail if it was not shocked. So, a node \vec{u} that was not shocked in the given solution can be shocked; such a node must exist since $\kappa < n$.

Note that, for any $e = (u_1, u_2, \dots, u_d) \in F$, if the d nodes $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d$ are shocked then \vec{e} fails at $t = 2$ was already shown in the proof of the completeness part. Thus, proof is complete if it is shown that, such a node \vec{e} does *not* fail at $t = 2$ if *at least* one of the nodes $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d$ is *not* shocked. Let $S \subset \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d\}$ be the set of shocked nodes among these d nodes. Then, \vec{e} does not fail at $t = 2$ since

$$\sum_{u_i \in S} \frac{\min\{\Phi(b_{\vec{u}_i} - \iota_{\vec{u}_i} + \mathcal{E}_{\vec{u}_i}) - \gamma(b_{\vec{u}_i} + \mathcal{E}_{\vec{u}_i}), b_{\vec{u}_i}\}}{\deg_{\text{in}}(\vec{u}_i)} \leq d - 1 \leq 0.995d = \gamma(b_{\vec{e}} + \mathcal{E}_{\vec{e}})$$

for all sufficiently large d . □

CHAPTER 3

EMPIRICAL EVALUATIONS OF GLOBAL STABILITY, MEASURES AND THEIR POLICY IMPLICATIONS

In this chapter, we use the model and the insolvency propagation equation described in the previous chapter, define a suitable global stability measure for this model, and perform a comprehensive evaluation of this stability measure over more than 700,000 combinations of networks types and parameter combinations. Based on our evaluations, we discover many interesting implications of our evaluations of this stability measure, and derive topological properties and parameter combinations that may be used to flag the network as a possible fragile network. The following interesting insights into the relationships of the stability with other relevant parameters of the network was discovered:

Effect of uneven distribution of assets: Networks where all banks have roughly the *same* external assets are more stable over similar networks in which fewer banks have a disproportionately higher external assets compared to the remaining banks, and failures of those banks with *higher* assets contribute more damage to the stability of the network. Furthermore, networks in which fewer banks have a disproportionately higher external assets, has a minimal stability even if their equity to asset ratio is large and comparable to loss of external assets. This is not the case for networks where all banks have roughly the same external assets. Thus, in summary, it can be concluded that banks with disproportionately large external assets (“banks that are too big”) affect the stability of the entire banking network in an *adverse* manner.

Effect of connectivity: For banking networks where all banks have roughly the same amount of external assets, higher connectivity leads to *lower* stability. In contrast, for banking networks in which few banks

have disproportionately higher external assets compared to the remaining banks, higher connectivity leads to *higher* global stability.

Correlated versus random failures: *Correlated* initial failures of banks causes more damage to the entire banking network as opposed to just *random* initial failures of banks.

Phase transition properties of global stability: The global stability exhibits several sharp *phase transitions* for various banking networks within certain parameter ranges.

3.1 Economic policy implications

Returning to the original motivation of flagging financial networks for potential vulnerabilities, the results suggest that a network model similar to that used here may be flagged for the following cases:

- the equity to asset ratios of most banks are low,
- the network has a highly skewed distribution of external assets and inter-bank exposures among its banks and the network is sufficiently sparse,
- the network does not have either a highly skewed distribution of external assets or a highly skewed distribution of inter-bank exposures among its banks, but the network is sufficiently dense.

3.2 Empirical measure of global stability

Let $\mathcal{K} \in (0, 1]$ be a real number¹ denoting the fraction of nodes in V that received the initial shock under a shocking mechanism Υ and let $S_{\Upsilon, \mathcal{K}}$ be the set of all possible $(\mathcal{K}n)$ -element subsets of V . The *vulnerability index*² of the network is then defined as³

¹ \mathcal{K} is a new parameter not used by prior researchers.

²Although simple topological properties such as clustering coefficients have been used to study properties of networks (79; 86), they are too simplistic for stability analysis of financial networks.

³In this definition, it is implicitly assumed that the shocking mechanism Υ allows one to select at least one set of $\mathcal{K}n$ nodes for the initial shock. Otherwise, ξ is defined to be zero.

$$\xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon) = \frac{1}{n} \times \mathbb{E} \left[\left| \lim_{t \rightarrow \infty} V_{\otimes \leftarrow} (t, V_{\text{shock}}) \right| : V_{\text{shock}} \text{ is selected randomly from } S_{\Upsilon, \mathcal{K}} \right]$$

In the above definition, the $\frac{1}{n}$ factor is only for a min-max normalization (55) to ensure that $0 \leq \xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon) \leq 1$.

Noting that no new node in the network may fail at a time $t \geq n$, the above expression for ξ can be simplified as:

$$\begin{aligned} \xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon) &= \frac{1}{n} \times \mathbb{E} \left[\left| V_{\otimes \leftarrow} (n, V_{\text{shock}}) \right| : V_{\text{shock}} \text{ is selected randomly from } S_{\Upsilon, \mathcal{K}} \right] \\ \implies \Pr \left[\left| V_{\otimes \leftarrow} (n, V_{\text{shock}}) \right| \geq n \xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon) : V_{\text{shock}} \text{ is selected randomly from } S_{\Upsilon, \mathcal{K}} \right] &> 0 \end{aligned}$$

As an example, $\xi(0.1, G, 0.3, 0.5, \text{random}) = 0.9$ means that with positive probability 90% nodes of the network G become insolvent with $\gamma = 0.3$ and $\Phi = 0.5$ if we provide an initial shock to a random subset of 10% of nodes of G . *Note that lower values of ξ imply higher global stability of a network.* For simplicity, the arguments of ξ can be omitted when they are clear from the context.

A pseudo-code for calculating ξ is shown in Figure 12.

```

(* initialization of parameters *)
 $t \leftarrow 0$  ;  $V_{\text{shock}}(0, V_{\text{shock}}) \leftarrow \emptyset$  ; continue = TRUE ;
 $V_{\text{shock}} \leftarrow$  set of  $\mathcal{K}$  nodes selected for initial insolvency based on shocking mechanism  $\Upsilon$ 
for every node  $v \in V$  do
   $\text{deg}_{\text{in}}(v, 0, V_{\text{shock}}) \leftarrow \text{deg}_{\text{in}}(v)$ 
endfor

(* provide an initial shock *)
for every node  $u \in V'$  do
   $c_u(0, V_{\text{shock}}) \leftarrow c_u - \Phi e_u$ 
endfor

(* loop until no new node fails *)
while ((continue = TRUE)  $\wedge$  ( $V_{\text{shock}}(t, V_{\text{shock}}) \neq V$ )) do
  for every node  $u \in V \setminus V_{\text{shock}}(t, V_{\text{shock}})$  do
     $c_u(t+1, V_{\text{shock}}) \leftarrow c_u(t, V_{\text{shock}})$ 
    for every node  $v \in V \setminus V_{\text{shock}}(t, V_{\text{shock}})$  do
      if ( $(c_v(t, V_{\text{shock}}) < 0) \wedge ((u, v) \in E)$ ) then (* propagate shock to in-neighbors *)
         $c_u(t+1, V_{\text{shock}}) \leftarrow c_u(t+1, V_{\text{shock}}) - \frac{\min\{|c_v(t, V_{\text{shock}})|, b_v\}}{\text{deg}_{\text{in}}(v, t, V_{\text{shock}})}$ 
      endfor
     $V_{\text{shock}}(t+1, V_{\text{shock}}) \leftarrow V_{\text{shock}}(t, V_{\text{shock}})$  ;
    if  $c_u(t, V_{\text{shock}}) < 0$  then  $V_{\text{shock}}(t+1, V_{\text{shock}}) \leftarrow V_{\text{shock}}(t+1, V_{\text{shock}}) \cup \{u\}$ 
      (* node  $u$  fails if its equity becomes negative *)
    endfor
   $t \leftarrow t + 1$ 
  if ( $V_{\text{shock}}(t, V_{\text{shock}}) = V_{\text{shock}}(t-1, V_{\text{shock}})$ ) then continue  $\leftarrow$  FALSE
  for every node  $u \in V \setminus V_{\text{shock}}(t, V_{\text{shock}})$  do
     $\text{deg}_{\text{in}}(u, t, V_{\text{shock}}) = \left| \left\{ v : ((v, u) \in E) \wedge (v \in V \setminus V_{\text{shock}}(t, V_{\text{shock}})) \right\} \right|$ 
    (* update in-degrees of alive nodes to exclude freshly dead nodes *)
  endfor
endwhile
 $\xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon) \leftarrow \frac{|V_{\text{shock}}(t, V_{\text{shock}})|}{n}$ 

```

Figure 12. Pseudo-code for calculating $\xi(\mathcal{K}, G, \gamma, \Phi, \Upsilon)$. Comments in the pseudo-code are enclosed by (* and *). An implementation of the pseudo-code is available at www2.cs.uic.edu/~dasgupta/financial-simulator-files.

parameter	explored values for the parameter	
network type	homogeneous	
	(α, β) -heterogeneous	$\alpha = 0.1, \beta = 0.95$ $\alpha = 0.2, \beta = 0.6$
network topology	directed scale-free	average degree 1 (in-arborescence)
		average degree 3 average degree 6
	directed Erdős-Rényi	average degree 3 average degree 6
shocking mechanism	idiosyncratic, coordinated	
number of nodes	50, 100, 300	
\mathcal{E}/\mathcal{I}	0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5	
Φ	0.5, 0.6, 0.7, 0.8, 0.9	
\mathcal{K}	0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9	
γ	0.05, 0.1, 0.15, \dots , $\Phi - 0.05$	

total
number of
parameter
combinations
> 700,000

TABLE IV. A summary of simulation environment and explored parameter space.

3.3 Simulation environment and explored parameter space

In Table IV a summary of the simulation environment and explored parameter space is provided. Individual components of the summary are discussed in Sections 3.3.1–3.3.4.

3.3.1 Network topology

One may obviously ask: *why not use “real” networks?* There are several obstacles however that make this desirable goal impossible to achieve. For example: **(a)** Due to their highly sensitive nature, such networks with all relevant parameters are *rarely* publicly available. **(b)** For the kind of inferences that is made here, hundreds of thousands of large networks is needed to have any statistical validity (here, more than 700,000 networks are explored).

Two topology models previously used by economists to generate random financial networks are considered:

- the *directed scale-free* (SF) network model (12) that has been used by prior financial network researchers such as (85; 77; 8; 32), and
- the *directed Erdős-Rényi* (ER) network model (20) that has been used by prior financial network researchers such as (84; 50; 71; 33; 25).

Generation of directed ER networks is computationally trivial: given a value $0 < p < 1$ that parameterizes the ER network, for every *ordered* pair of distinct nodes (u, v) set $\Pr[(u, v) \in E] = 1/p$. Letting $p = d/n$ generates a random ER network whose average degree is d with high probability.

The directed SF networks used here are generated using the algorithm outlined by Bollobas et al. (20).

The algorithm works as follows. Let $a, b, \eta, \delta_{\text{in}}$ (in-degree) and δ_{out} (out-degree)

be non-negative real numbers with $a + b + \eta = 1$. The initial graph $G(0)$ at step $\ell = 0$ has just one node with no edges. At step $\ell > 0$ the graph $G(\ell)$ has exactly ℓ edges and a random number n_ℓ of nodes. For $\ell \geq 0$, $G(\ell + 1)$ is obtained from $G(\ell)$ by using the following rules:

- With probability a , add a new node v together with an edge from v to an existing node w , where w is chosen randomly such that

$$\Pr[w = u] = \frac{(d_{\text{in}}(u) + \delta_{\text{in}})}{(\ell + \delta_{\text{in}} n_\ell)}$$

for every existing node u , where $d_{\text{in}}(u)$ is the in-degree of node u in $G(\ell)$.

- With probability b , add an edge from an existing node v to an existing node w , where v and w are chosen independently, such that

$$\Pr[v = u] = \frac{d_{\text{out}}(u) + \delta_{\text{out}}}{\ell + \delta_{\text{out}} n_\ell} \text{ for every existing node } u$$

$$\Pr[w = u] = \frac{d_{\text{in}}(u) + \delta_{\text{in}}}{\ell + \delta_{\text{in}} n_\ell} \text{ for every existing node } u$$

where $d_{\text{out}}(u)$ is the out-degree of node u in $G(\ell)$.

- With probability η , add a new node w and an edge from an existing node v to w , where v is chosen such that $\Pr[v=u] = \frac{d_{\text{out}}(u) + \delta_{\text{out}}}{\ell + \delta_{\text{out}} n_\ell}$ for every existing node u .

The *effect of connectivity* on network stability is studied by generating random SF and ER networks with average degrees¹ of 3 and 6.

In addition, to study the effect of sparse hierarchical topology on network stability, the Barabasi-Albert *preferential-attachment* model (12) is used to generate random *in-arborescence* networks. In-arborescences are directed rooted trees with all edges oriented towards the root (see Figure 13), and have the following well-known topological properties:

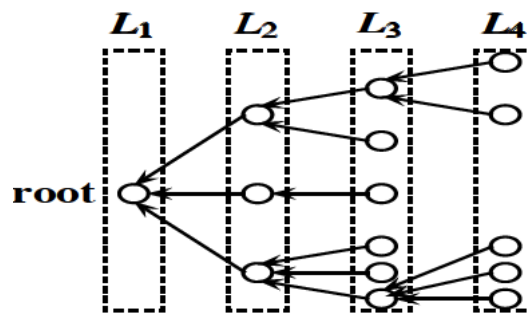


Figure 13. An in-arborescence graph.

- They belong to the class of *sparsest connected directed acyclic* graphs.
- They are *hierarchical* networks, *i.e.*, the nodes can be partitioned into levels L_1, L_2, \dots, L_p such that L_1 has exactly one node (the “root”) and nodes in any level L_i have directed edges only to nodes in L_{i-1} (see Figure 13). The root may model a “central bank” that lends to other banks but does not borrow from any bank.

¹There are many ways to fix the parameters to get the desired average degree. For example, as observed in (20), letting $\delta_{\text{out}} = 0$ and $\alpha > 0$, one obtains $\mathbb{E}[\text{number of nodes in } G(t) \text{ of in-degree } x] \propto x^{-(1 + \frac{1+\alpha_{\text{in}}(\alpha+\eta)}{\alpha+\beta})} t$ and $\mathbb{E}[\text{number of nodes in } G(t) \text{ of out-degree } x] \propto x^{-(\frac{2-\alpha}{1-\alpha})} t$.

The algorithm for generating a random in-arborescence network G using the preferential-attachment model (12) is as follows:

- Initialize $G = (V, E)$ to have one node (the root) and no edges.
- Repeat the following steps till G has n nodes:
 - Randomly select a node u in G such that, for every node v in G , $\Pr[u = v] = \frac{\deg(v)}{\sum_{w \in V} \deg(w)}$ where $\deg(y)$ denotes the degree of node y in G .
 - Add a new node x and an undirected edge $\{x, u\}$ in G .
- Orient all the edges towards the root.

3.3.2 Shocking mechanisms Υ

Recall that a shocking mechanism Υ provides a rule to select the initial subset of nodes to be shocked.

The following two mechanisms are used to select the nodes to receive the initial shock.

Idiosyncratic (random) shocking mechanism A subset of nodes *uniformly at random* is selected. This corresponds to random idiosyncratic initial insolvencies of banks, and is a choice that has been used by prior researchers such as (78; 56; 54; 50; 72).

Coordinated shocking mechanism¹. In this type of *non-idiosyncratic* correlated shocking mechanism, an *adversarial* role² is played in selecting nodes for the initial shock that may cause more damage to the stability

¹While correlated shocking mechanisms affecting a correlated subset of banks are relevant in practice, prior researchers such as (78; 54; 56; 50; 72) have mostly used idiosyncratic shocking mechanisms. There are at least two reasons for this. Firstly, idiosyncratic shocks are a *cleaner* way to study the stability of the topology of the banking network. Secondly, it is not *a priori* clear what kind of correlations in the shocking mechanism would lead to failure of more nodes than idiosyncratic shocks in a *statistically significant way*. The coordinated shocking mechanism intuitively corresponds shocks in which banks that are “too big to fail” in terms of their assets are correlated. The conclusion ⑥ shows that coordinated shocks do indeed cause more statistically significant damage to the stability of the network as opposed to random shocks.

²Usage of adversarial strategies in measuring the worst-case bounds for network properties are very common in the algorithmic literature; see, for example, see the book (21).

of the network. The selection of an adversarial strategy depends on whether the network is homogeneous or heterogeneous. The coordinated shocking mechanism falls under the general category of *correlated* shocks where the nodes with high (weighted) in-degrees are correlated.

For homogeneous networks, recall that all nodes have the same share of the total external asset \mathcal{E} . However, the total interbank exposure b_v of a node v is directly proportional to the in-degree of v , and, as per Equation (Equation 2.1), nodes with higher inter-bank exposures are more likely to transmit the shock to a larger number of *other* nodes. Thus, an adversarial role is played by selecting a set of $\mathcal{K}n$ nodes in *non-increasing* order of their in-degrees starting from a node with the highest in-degree.

For heterogeneous networks, nodes with higher “weighted” in-degrees (*i.e.*, with higher values of the sum of *weights* of incoming edges) represent nodes that have larger external assets than other nodes, and have more inter-bank exposures. Thus, for heterogeneous networks an adversarial role is played by selecting $\mathcal{K}n$ nodes in non-increasing order of their *weighted in-degrees* starting from a node with the highest weighted in-degree.

3.3.3 Network type: (α, β) -heterogeneous networks

Recall that in a heterogeneous network it is possible to have a few banks whose external assets or interbank exposures are *significantly larger* than the rest of the banks, *i.e.*, it is possible to have a few banks that are “too big”, and thus heterogeneous networks permit investigation of the effect of such big banks on the global stability of the entire network. To this end, a (α, β) -heterogeneous network is defined as follows.

Definition 3.3.1. [(α, β) -heterogeneous network] Let $\tilde{V} \subseteq V$ be a random subset V of αn nodes and let \tilde{E} be the set of edges that have at least one end-point from \tilde{V} . For $0 < \alpha, \beta < 1$, a (α, β) -heterogeneous network $G = (V, E)$ is one in which the total external and internal assets are distributed in the following manner:

Distribution of \mathcal{E} :

- distribute $\beta\mathcal{E}$ part of the total external asset \mathcal{E} equally among the αn nodes in \tilde{V} , and

- distribute the remaining part $(1 - \beta)\mathcal{E}$ of \mathcal{E} equally among the remaining $(1 - \alpha)n$ nodes.

Distribution of \mathcal{I} :

- Distribute $\beta\mathcal{I}$ part of the total interbank exposure \mathcal{I} equally among a random subset of $\alpha|\tilde{E}|$ of edges from the edges in \tilde{E} , and
- distribute the remaining part $(1 - \beta)\mathcal{I}$ of \mathcal{I} equally among the remaining $|E| - \alpha|\tilde{E}|$ edges.

The simulations for (α, β) -heterogeneous networks was performed for $(\alpha, \beta) = (0.1, 0.95)$ and $(\alpha, \beta) = (0.2, 0.6)$. The combination $(\alpha, \beta) = (0.1, 0.95)$ corresponds to the extreme situation in which 95% of the assets and exposures involve 10% of banks, thus creating a minority of banks that are significantly larger than the remaining banks. The other combination $(\alpha, \beta) = (0.2, 0.6)$ corresponds to a less extreme situation in which there are a larger number of moderately large banks.

3.3.4 Other minor details

To correct statistical biases, for each combinations of parameters, shock types and network types, 10 corresponding random networks was generated and computed the average value of the vulnerability index over these 10 runs. For ER and SF random networks, the values of network generation parameters are selected such that the expected number of edges of the network is $3n$ or $6n$ depending on whether we require the average degree of the network to be 3 or 6, respectively.

The minimum difference between two non-identical values of the average vulnerability index over 10 runs for two n -node networks is $1/(10n)$. Thus, to allow for minor statistical biases introduced by any random graph generation method, two vulnerability indices are considered to be same (within the margin of statistical error) if their absolute difference is no more than $1/(3n)$, which is above $1/(10n)$ but no more than 0.7% of the total range of the vulnerability indices.

Finally, it can be assumed without loss of generality that $\mathcal{I} = m$, since otherwise if $\mu = \frac{\mathcal{I}}{m} \neq 1$ then each of the quantities ι_v , b_v and \mathcal{E} can be divided by μ without changing the outcome of the insolvency propagation procedure.

3.4 Results

In this section, many interesting relationships of the stability with other relevant parameters of the network based on the comprehensive evaluation and analysis of this stability measure are discussed.

It is easy to see that there are many (at least several thousands, but significantly more in most cases) networks in the original sets of networks that are compared in two different scenarios in Table V – Table XXII, thereby assuring the statistical validity of the comparison results.

3.4.1 Effect of unequal distribution of total assets \mathcal{E} and \mathcal{I}

As the analysis shows, nodes with disproportionately large external assets affect the stability of the entire network in an *adverse* manner, and more uneven distribution of assets among nodes in the network makes the network less stable.

3.4.1.1 Effect on global stability

For the same value of the common parameters n , $\frac{\mathcal{E}}{\mathcal{I}}$, \mathcal{K} , Φ and γ , for the same network type (ER, SF or in-arborescence) of same average degree (6, 3 or 1) and for the same shocking mechanism Υ (coordinated or idiosyncratic), the value of ξ for the homogeneous network with the corresponding values of ξ for (0.1, 0.95)-heterogeneous and (0.2, 0.6)-heterogeneous networks is compared. The comparison results shown in Table V show most of the entries as being at least 90%. Thus, it can be concluded as:

- ① *networks with all nodes having the same external assets display higher stability over similar networks with fewer nodes having disproportionately higher external assets.*

TABLE V. Comparison of stabilities of (α, β) -heterogeneous networks with their homogeneous counter-parts over all parameter ranges. The numbers are the percentages of data points for which $\xi_{(\alpha, \beta)\text{-heterogeneous}}$ was at least $\xi_{\text{homogeneous}}$.

	In-arborescence		ER average degree 3		ER average degree 6		SF average degree 3		SF average degree 6	
	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$
	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$
coordinated shock	66.91%	60.22%	99.26%	98.91%	98.46%	98.00%	98.22%	91.68%	99.13%	97.4%
idiosyncratic shock	92.75%	81.79%	97.76%	96.81%	98.16%	97.61%	98.86%	94.84%	98.83%	97.22%

Formal intuition behind the conclusion in ①

In spite of the highly non-linear nature of Equation (Equation 2.1), the following formal intuition may help to explain the conclusion in ①.

Lemma 3.4.1. Fix $\gamma, \Phi, \mathcal{E}, \mathcal{I}$ and the graph G . Consider any node $v \in V_{\text{shock}}$ and suppose that v fails due to the initial shock. For every edge $(u, v) \in E$, let $\Delta_{\text{homo}}(u)$ and $\Delta_{\text{hetero}}(u)$ be the amount of shock received by node u at time $t = 1$ if G is homogeneous or heterogeneous, respectively. Then,

$$\mathbb{E}[\Delta_{\text{hetero}}(u)] \geq \frac{\beta}{\alpha} \mathbb{E}[\Delta_{\text{homo}}(u)] = \begin{cases} 9.5 \mathbb{E}[\Delta_{\text{homo}}(u)], & \text{if } (\alpha, \beta) = (0.1, 0.95) \\ 3 \mathbb{E}[\Delta_{\text{homo}}(u)], & \text{if } (\alpha, \beta) = (0.2, 0.6) \end{cases}$$

Lemma 3.4.1 implies that $\mathbb{E}[\Delta_{\text{hetero}}(u)]$ is much bigger than $\mathbb{E}[\Delta_{\text{homo}}(u)]$, and thus more nodes are likely to fail beyond $t > 0$ leading to a lower stability for heterogeneous networks.

Proof of Lemma 3.4.1

The notations in Definition 3.3.1 is reused. Using Equation (Equation 2.1), for every edge $(u, v) \in E$, the amount of shock received by node u at time $t = 1$ is as follows:

- If G is homogeneous then

$$\begin{aligned} \mathbb{E}[\Delta_{\text{homo}}(u)] &= \frac{\min\left\{\Phi\left(\text{deg}_{\text{in}}(v) - \text{deg}_{\text{out}}(v) + \frac{\mathcal{E}}{n}\right) - \gamma\left(\text{deg}_{\text{in}}(v) + \frac{\mathcal{E}}{n}\right), \text{deg}_{\text{in}}(v)\right\}}{\text{deg}_{\text{in}}(v)} \\ &= \min\left\{\left(\Phi - \gamma\right) + \frac{\frac{\mathcal{E}}{n}(\Phi - \gamma) - \Phi \text{deg}_{\text{out}}(v)}{\text{deg}_{\text{in}}(v)}, 1\right\} \end{aligned}$$

- If G is (α, β) -heterogeneous, then $\sigma_v = \frac{\beta}{\alpha}$ and, using linearity of expectation, we get

$$\begin{aligned} \mathbb{E}[b_v] &= \mathbb{E}\left[\sum_{(u,v) \in \tilde{E}} \left(\alpha \frac{\beta \mathcal{J}}{\alpha |\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] \\ &= \text{deg}_{\text{in}}(v) \mathbb{E}\left[\left(\beta \frac{\mathcal{J}}{|\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] \\ \mathbb{E}[b_v - \iota_v] &= \mathbb{E}\left[\sum_{(u,v) \in \tilde{E}} \left(\alpha \frac{\beta \mathcal{J}}{\alpha |\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right) - \sum_{(v,u) \in \tilde{E}} \left(\alpha \frac{\beta \mathcal{J}}{\alpha |\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] \\ &= \sum_{(u,v) \in \tilde{E}} \mathbb{E}\left[\left(\beta \frac{\mathcal{J}}{|\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] - \sum_{(v,u) \in \tilde{E}} \mathbb{E}\left[\left(\beta \frac{\mathcal{J}}{|\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] \\ &= (\text{deg}_{\text{in}}(v) - \text{deg}_{\text{out}}(v)) \mathbb{E}\left[\left(\beta \frac{\mathcal{J}}{|\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] \\ \mathbb{E}[\Delta_{\text{hetero}}(u)] &= \mathbb{E}\left[\frac{\min\left\{\Phi(b_v - \iota_v + \sigma_v \mathcal{E}) - \gamma(b_v + \sigma_v \mathcal{E}), b_v\right\}}{\text{deg}_{\text{in}}(v)}\right] \\ &= \min\left\{\mathbb{E}\left[\frac{\Phi(b_v - \iota_v + \sigma_v \mathcal{E}) - \gamma(b_v + \sigma_v \mathcal{E})}{\text{deg}_{\text{in}}(v)}\right], \mathbb{E}\left[\frac{b_v}{\text{deg}_{\text{in}}(v)}\right]\right\} \\ &= \min\left\{\frac{\Phi}{\text{deg}_{\text{in}}(v)} \mathbb{E}[b_v - \iota_v] + \frac{\Phi \sigma_v \mathcal{E}}{\text{deg}_{\text{in}}(v)} - \frac{\gamma}{\text{deg}_{\text{in}}(v)} \mathbb{E}[b_v] - \frac{\gamma \sigma_v \mathcal{E}}{\text{deg}_{\text{in}}(v)}, \frac{1}{\text{deg}_{\text{in}}(v)} \mathbb{E}[b_v]\right\} \\ &= \min\left\{\frac{\Phi}{\text{deg}_{\text{in}}(v)} (\text{deg}_{\text{in}}(v) - \text{deg}_{\text{out}}(v)) \mathbb{E}\left[\left(\beta \frac{\mathcal{J}}{|\tilde{E}|} + (1 - \alpha) \frac{(1 - \beta) \mathcal{J}}{|E| - \alpha |\tilde{E}|}\right)\right] + \frac{\Phi \sigma_v \mathcal{E}}{\text{deg}_{\text{in}}(v)}\right\} \end{aligned}$$

$$\begin{aligned}
& - \frac{\gamma}{\deg_{\text{in}}(v)} \deg_{\text{in}}(v) \mathbb{E} \left[\left(\beta \frac{\mathcal{L}}{|\bar{E}|} + (1 - \alpha) \frac{(1-\beta)\mathcal{L}}{|\bar{E}| - \alpha|\bar{E}|} \right) \right] - \frac{\gamma \sigma_v \mathcal{E}}{\deg_{\text{in}}(v)}, \\
& \mathbb{E} \left[\left(\beta \frac{\mathcal{L}}{|\bar{E}|} + (1 - \alpha) \frac{(1-\beta)\mathcal{L}}{|\bar{E}| - \alpha|\bar{E}|} \right) \right] \Big\} \\
& = \min \left\{ \frac{\Phi(\deg_{\text{in}}(v) - \deg_{\text{out}}(v)) - \gamma \deg_{\text{in}}(v)}{\deg_{\text{in}}(v)} \mathbb{E} \left[\left(\beta \frac{\mathcal{L}}{|\bar{E}|} + (1 - \alpha) \frac{(1-\beta)\mathcal{L}}{|\bar{E}| - \alpha|\bar{E}|} \right) \right] + \frac{(\Phi - \gamma) \sigma_v \mathcal{E}}{\deg_{\text{in}}(v)}, \right. \\
& \quad \left. \mathbb{E} \left[\left(\beta \frac{\mathcal{L}}{|\bar{E}|} + (1 - \alpha) \frac{(1-\beta)\mathcal{L}}{|\bar{E}| - \alpha|\bar{E}|} \right) \right] \right\} \\
& \geq \min \left\{ \frac{\Phi(\deg_{\text{in}}(v) - \deg_{\text{out}}(v)) - \gamma \deg_{\text{in}}(v)}{\deg_{\text{in}}(v)} \beta \mathbb{E} \left[\frac{\mathcal{L}}{|\bar{E}|} \right] + \frac{(\Phi - \gamma) \sigma_v \mathcal{E}}{\deg_{\text{in}}(v)}, \beta \mathbb{E} \left[\frac{\mathcal{L}}{|\bar{E}|} \right] \right\} \\
& \geq \min \left\{ \left(\frac{\Phi(\deg_{\text{in}}(v) - \deg_{\text{out}}(v)) - \gamma \deg_{\text{in}}(v)}{\deg_{\text{in}}(v)} \right) \left(\frac{\beta}{\alpha} \right) + \left(\frac{\beta}{\alpha} \right) \left(\frac{(\Phi - \gamma) \mathcal{E}}{\deg_{\text{in}}(v)} \right), \frac{\beta}{\alpha} \right\} \\
& \quad \text{since } \mathbb{E} \left[\frac{\mathcal{L}}{|\bar{E}|} \right] = \mathbb{E} \left[\frac{|E|}{|\bar{E}|} \right] = \alpha \\
& \geq \frac{\beta}{\alpha} \min \left\{ (\Phi - \gamma) + \frac{(\Phi - \gamma) \mathcal{E} - \Phi \deg_{\text{out}}(v)}{\deg_{\text{in}}(v)}, 1 \right\} \\
& \geq \frac{\beta}{\alpha} \mathbb{E} [\Delta_{\text{homo}}(u)]
\end{aligned}$$

3.4.1.2 Effect on residual instability

For homogeneous networks, if the equity to asset ratio γ is close enough to the severity of the shock Φ then the network almost always tends to be perfectly stable, as one would intuitively expect. However, the above property is *not* true in general for highly heterogeneous networks in the sense that, even when γ is close to Φ , these networks (irrespective of their topologies and densities) have a *minimum* amount of global instability (which is termed as the *residual instability*)¹.

In Table VI –Table XV residual instabilities for different types of homogeneous and heterogeneous networks under coordinated and idiosyncratic shocks are tabulated. The numbers in these tables show, for each combination of network types, $|V|$, shocking mechanism and values of Φ and γ such that $|\Phi - \gamma| = 0.05$,

¹For visual illustrations to this phenomena, see Figure 14–Figure 16. For example, in Figure 14, when γ is 45% and Φ is only 5% more than γ , the vulnerability index ξ is approximately 0 for *all* the 9 combinations of parameters, but in Figure 15–Figure 16 *all* the 18 networks have a value of $\xi \geq 0.1$ even when γ is 45% and the severity of the shock is only 5% more than γ .

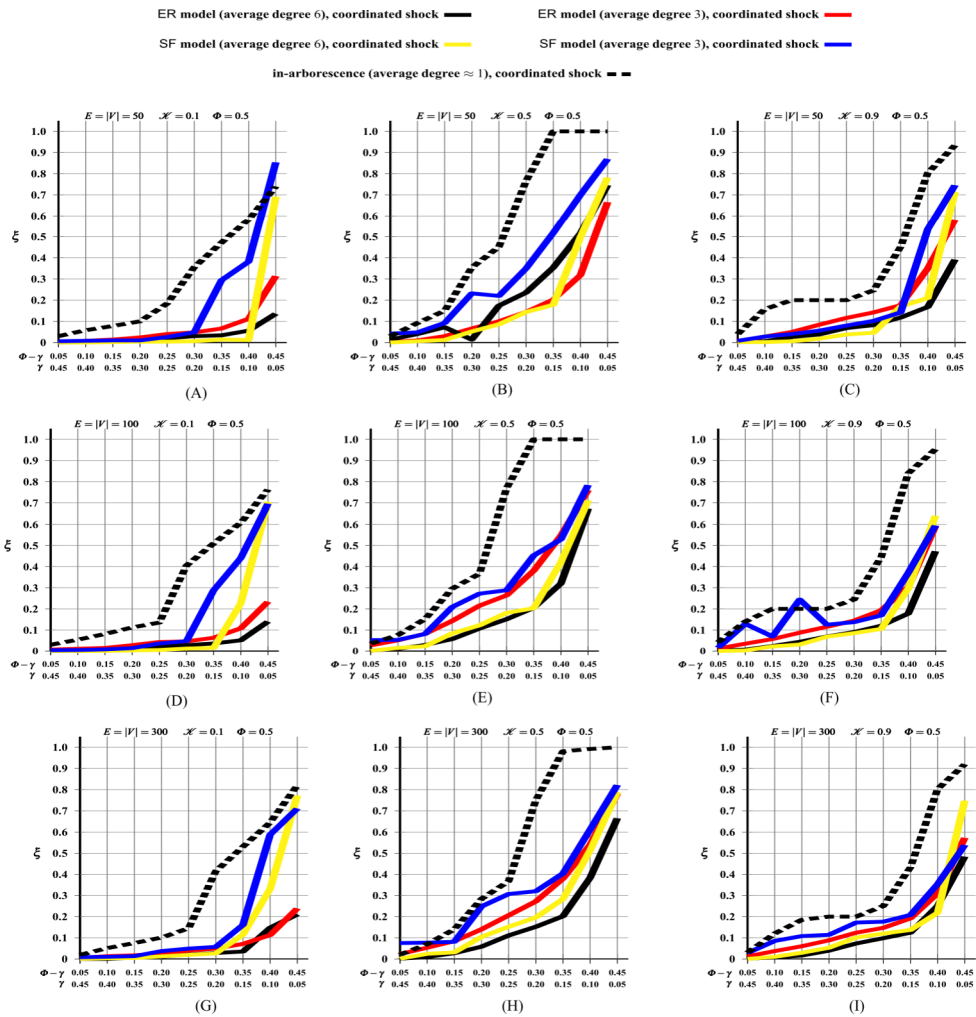


Figure 14. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for homogeneous networks. Lower values of ξ imply higher global stability of a network.

the percentage of networks with this combination for which the vulnerability index ξ was less than 0.05, 0.1 or 0.2. As the reader will observe, all the numbers for heterogeneous networks are significantly lower than their homogeneous counter-parts. Thus, it can be concluded as:

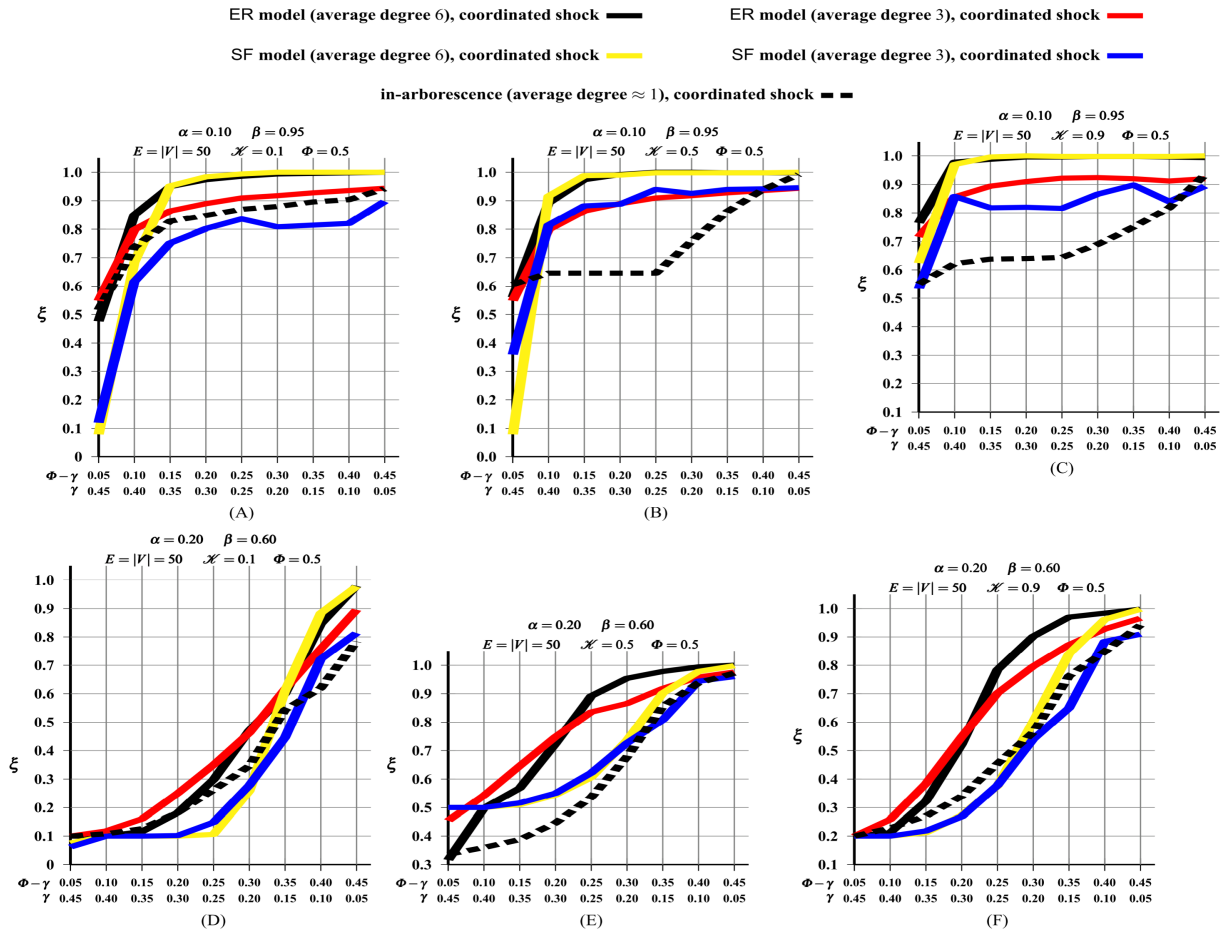


Figure 15. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

② a heterogeneous network, in contrast to its corresponding homogeneous version, has a residual minimum instability even if its equity to asset ratio is very large and close to the severity of the shock.

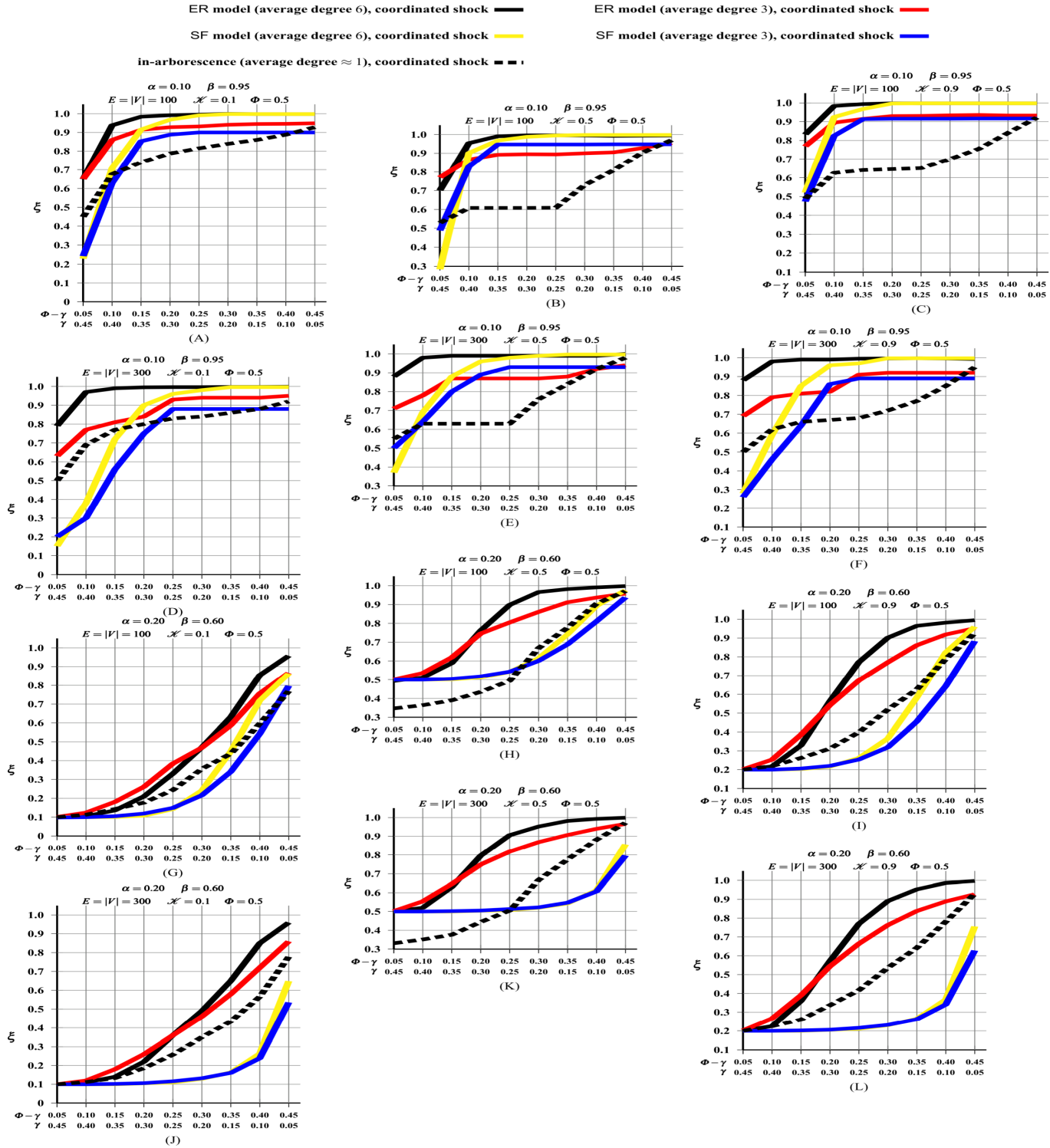


Figure 16. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

TABLE VI. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		$\Phi = 0.5, \gamma = 0.45$			coordinated shock $\Phi = 0.5, \gamma = 0.40$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	73%	73%	73%	0%	31%	59%
		ER, average degree 3	89%	100%	100%	43%	84%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	44%	84%	100%	25%	57%	88%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
V = 50	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	0%	1%	0%	0%	0%
		ER, average degree 6	8%	9%	10%	2%	6%	6%
		SF, average degree 3	2%	6%	15%	0%	2%	5%
		SF, average degree 6	18%	23%	30%	9%	10%	11%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	4%	7%	19%	2%	6%	16%
		ER, average degree 6	8%	12%	24%	6%	7%	16%
		SF, average degree 3	2%	6%	22%	0%	2%	18%
		SF, average degree 6	8%	12%	24%	7%	8%	16%
	homogeneous	in-arborescence	73%	73%	73%	0%	34%	73%
		ER, average degree 3	66%	100%	100%	25%	64%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	29%	61%	100%	20%	42%	83%
		SF, average degree 6	100%	100%	100%	90%	100%	100%
V = 100	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	0%	6%	0%	0%	1%
		ER, average degree 6	6%	6%	6%	4%	6%	6%
		SF, average degree 3	0%	0%	7%	0%	0%	3%
		SF, average degree 6	6%	10%	15%	6%	6%	6%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	0%	6%	16%	0%	4%	16%
		ER, average degree 6	6%	7%	16%	6%	6%	16%
		SF, average degree 3	0%	2%	14%	0%	1%	13%
		SF, average degree 6	7%	8%	17%	6%	7%	16%
	homogeneous	in-arborescence	73%	73%	73%	0%	55%	73%
		ER, average degree 3	71%	97%	100%	22%	60%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	22%	44%	86%	18%	36%	74%
		SF, average degree 6	100%	100%	100%	88%	100%	100%
V = 300	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	0%	6%	0%	0%	1%
		ER, average degree 6	6%	6%	6%	0%	6%	6%
		SF, average degree 3	0%	0%	10%	0%	0%	2%
		SF, average degree 6	6%	6%	16%	6%	6%	6%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	0%	6%	16%	0%	4%	16%
		ER, average degree 6	6%	6%	16%	6%	6%	16%
		SF, average degree 3	0%	0%	13%	0%	0%	12%
		SF, average degree 6	6%	7%	16%	6%	6%	16%

TABLE VII. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		coordinated shock						
		$\Phi = 0.6, \gamma = 0.55$			$\Phi = 0.6, \gamma = 0.50$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	93%	93%	93%	2%	69%	93%
		ER, average degree 3	97%	100%	100%	64%	95%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	56%	96%	100%	44%	87%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
V = 50	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	1%	0%	0%	0%
		ER, average degree 3	0%	0%	5%	0%	0%	1%
		ER, average degree 6	9%	11%	14%	5%	7%	7%
		SF, average degree 3	5%	9%	19%	0%	3%	8%
		SF, average degree 6	22%	28%	41%	10%	15%	17%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	5%	7%	20%	3%	7%	16%
		ER, average degree 6	11%	14%	26%	7%	7%	19%
		SF, average degree 3	2%	8%	23%	1%	3%	18%
		SF, average degree 6	9%	15%	26%	7%	8%	18%
	homogeneous	in-arborescence	93%	93%	93%	23%	84%	93%
		ER, average degree 3	85%	100%	100%	47%	86%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	37%	75%	100%	35%	71%	100%
		SF, average degree 6	100%	100%	100%	96%	100%	100%
V = 100	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	3%
		ER, average degree 6	7%	7%	9%	7%	7%	7%
		SF, average degree 3	0%	1%	9%	0%	0%	5%
		SF, average degree 6	10%	15%	21%	7%	7%	11%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	0%	7%	17%	0%	5%	16%
		ER, average degree 6	7%	7%	19%	7%	7%	16%
		SF, average degree 3	1%	3%	15%	0%	1%	13%
		SF, average degree 6	7%	9%	18%	7%	7%	17%
	homogeneous	in-arborescence	93%	93%	93%	79%	93%	93%
		ER, average degree 3	93%	100%	100%	48%	85%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	28%	56%	97%	28%	56%	97%
		SF, average degree 6	100%	100%	100%	93%	100%	100%
V = 300	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	3%
		ER, average degree 6	7%	7%	7%	3%	7%	7%
		SF, average degree 3	0%	0%	11%	0%	0%	3%
		SF, average degree 6	7%	7%	18%	7%	7%	7%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	1%	7%	17%	1%	5%	16%
		ER, average degree 6	7%	7%	16%	7%	7%	16%
		SF, average degree 3	0%	1%	13%	0%	0%	13%
		SF, average degree 6	7%	7%	16%	7%	7%	16%

TABLE VIII. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		coordinated shock						
		$\Phi = 0.7, \gamma = 0.65$			$\Phi = 0.7, \gamma = 0.60$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	93%	93%	93%	35%	88%	93%
		ER, average degree 3	97%	100%	100%	81%	100%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	56%	96%	100%	52%	96%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	1%	7%	0%	0%	0%
		ER, average degree 3	0%	0%	5%	0%	0%	1%
		ER, average degree 6	10%	13%	16%	7%	7%	9%
		SF, average degree 3	7%	12%	25%	1%	4%	9%
		SF, average degree 6	26%	32%	49%	14%	15%	21%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	10%	0%	0%	9%
		ER, average degree 3	5%	8%	21%	3%	7%	17%
		ER, average degree 6	13%	18%	30%	7%	7%	20%
		SF, average degree 3	3%	9%	25%	1%	3%	19%
		SF, average degree 6	9%	16%	28%	8%	9%	19%
	homogeneous	in-arborescence	93%	93%	93%	31%	93%	93%
		ER, average degree 3	85%	100%	100%	62%	97%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	37%	75%	100%	37%	75%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	5%
		ER, average degree 6	7%	7%	9%	7%	7%	7%
		SF, average degree 3	0%	2%	15%	0%	0%	5%
		SF, average degree 6	11%	15%	27%	7%	7%	11%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	0%	7%	18%	0%	7%	16%
		ER, average degree 6	7%	7%	19%	7%	7%	16%
		SF, average degree 3	1%	4%	17%	0%	2%	13%
		SF, average degree 6	8%	10%	19%	7%	8%	17%
	homogeneous	in-arborescence	93%	93%	93%	56%	93%	93%
		ER, average degree 3	93%	100%	100%	65%	97%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	28%	56%	97%	28%	56%	97%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 300$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	4%
		ER, average degree 6	7%	7%	7%	6%	7%	7%
		SF, average degree 3	0%	0%	15%	0%	0%	3%
		SF, average degree 6	7%	7%	21%	7%	7%	7%
(0.2, 0.6)-heterogeneous		in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	1%	7%	17%	0%	7%	16%
		ER, average degree 6	7%	7%	16%	7%	7%	16%
		SF, average degree 3	0%	1%	14%	0%	0%	13%
		SF, average degree 6	7%	8%	16%	7%	7%	16%

TABLE IX. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		coordinate shock					
		$\Phi = 0.8, \gamma = 0.75$			$\Phi = 0.8, \gamma = 0.70$		
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$
	in-arborescence	0%	0%	93%	0%	0%	93%
	ER, average degree 3	97%	100%	100%	89%	100%	100%
	ER, average degree 6	100%	100%	100%	100%	100%	100%
	SF, average degree 3	56%	96%	100%	54%	96%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	in-arborescence	1%	7%	7%	0%	0%	0%
	ER, average degree 3	0%	1%	6%	0%	0%	1%
	ER, average degree 6	12%	15%	19%	7%	7%	9%
	SF, average degree 3	9%	15%	32%	1%	5%	11%
	SF, average degree 6	27%	37%	60%	14%	19%	21%
(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	10%	0%	0%	9%
	ER, average degree 3	5%	11%	22%	4%	7%	17%
	ER, average degree 6	14%	19%	30%	7%	11%	21%
	SF, average degree 3	3%	11%	27%	1%	4%	19%
	SF, average degree 6	11%	19%	31%	8%	9%	21%
	in-arborescence	93%	93%	93%	51%	93%	93%
	ER, average degree 3	85%	100%	100%	71%	100%	100%
	ER, average degree 6	100%	100%	100%	100%	100%	100%
	SF, average degree 3	37%	75%	100%	37%	75%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$	in-arborescence	0%	0%	1%	0%	0%	0%
	ER, average degree 3	0%	1%	7%	0%	0%	5%
	ER, average degree 6	7%	9%	10%	7%	7%	7%
	SF, average degree 3	0%	3%	20%	0%	0%	6%
	SF, average degree 6	11%	19%	33%	7%	11%	11%
(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
	ER, average degree 3	0%	7%	19%	0%	7%	17%
	ER, average degree 6	7%	8%	21%	7%	7%	17%
	SF, average degree 3	1%	5%	18%	0%	2%	14%
	SF, average degree 6	8%	10%	23%	7%	8%	17%
	in-arborescence	93%	93%	93%	76%	93%	93%
	ER, average degree 3	93%	100%	100%	76%	99%	100%
	ER, average degree 6	100%	100%	100%	100%	100%	100%
	SF, average degree 3	28%	56%	97%	28%	56%	97%
	SF, average degree 6	100%	100%	100%	99%	100%	100%
$ V = 300$	in-arborescence	0%	0%	0%	0%	0%	0%
	ER, average degree 3	0%	1%	7%	0%	1%	5%
	ER, average degree 6	7%	7%	7%	7%	7%	7%
	SF, average degree 3	0%	1%	16%	0%	0%	6%
	SF, average degree 6	7%	11%	22%	7%	7%	8%
(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
	ER, average degree 3	1%	7%	17%	1%	7%	16%
	ER, average degree 6	7%	7%	16%	7%	7%	16%
	SF, average degree 3	0%	1%	14%	0%	1%	13%
	SF, average degree 6	7%	8%	17%	7%	7%	16%

TABLE X. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		coordinated shock						
		$\Phi = 0.9, \gamma = 0.85$			$\Phi = 0.9, \gamma = 0.80$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	93%	93%	93%	81%	93%	93%
		ER, average degree 3	98%	100%	100%	95%	100%	100%
		ER, average degree 6	0%	0%	0%	100%	100%	100%
		SF, average degree 3	56%	96%	100%	56%	96%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	(0.1, 0.95)-heterogeneous	in-arborescence	7%	7%	7%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	1%
		ER, average degree 6	13%	18%	22%	7%	9%	10%
		SF, average degree 3	11%	17%	37%	3%	5%	13%
		SF, average degree 6	31%	41%	66%	18%	19%	25%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	0%	15%	0%	0%	9%
		ER, average degree 3	5%	11%	25%	5%	7%	19%
		ER, average degree 6	14%	19%	34%	7%	12%	23%
		SF, average degree 3	4%	12%	28%	2%	6%	21%
		SF, average degree 6	17%	21%	33%	8%	11%	23%
	homogeneous	in-arborescence	93%	93%	93%	75%	93%	93%
		ER, average degree 3	85%	100%	100%	78%	100%	100%
		ER, average degree 6	0%	0%	0%	100%	100%	100%
		SF, average degree 3	37%	75%	100%	37%	75%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	1%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	0%	6%
		ER, average degree 6	7%	9%	10%	7%	7%	7%
		SF, average degree 3	1%	4%	25%	0%	1%	7%
		SF, average degree 6	14%	22%	41%	7%	11%	15%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	0%	7%	21%	0%	7%	17%
		ER, average degree 6	7%	8%	23%	7%	7%	17%
		SF, average degree 3	1%	5%	19%	1%	3%	15%
		SF, average degree 6	9%	11%	23%	7%	9%	17%
	homogeneous	in-arborescence	93%	93%	93%	93%	93%	93%
		ER, average degree 3	0%	0%	100%	85%	99%	100%
		ER, average degree 6	0%	0%	0%	100%	100%	100%
		SF, average degree 3	28%	56%	97%	28%	56%	97%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 300$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	0%	0%	0%	0%	0%
		ER, average degree 3	0%	1%	7%	0%	1%	6%
		ER, average degree 6	7%	7%	7%	7%	7%	7%
		SF, average degree 3	0%	1%	17%	0%	0%	10%
		SF, average degree 6	7%	11%	24%	7%	7%	10%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	1%	7%	17%	1%	7%	16%
		ER, average degree 6	7%	7%	17%	7%	7%	16%
		SF, average degree 3	0%	1%	14%	0%	1%	13%
		SF, average degree 6	7%	8%	17%	7%	7%	16%

TABLE XI. Residual instabilities of homogeneous versus heterogeneous networks under idiosyncratic shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		idiosyncratic shock						
		$\Phi = 0.5, \gamma = 0.45$			$\Phi = 0.5, \gamma = 0.40$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	76%	78%	82%	26%	58%	75%
		ER, average degree 3	99%	100%	100%	43%	84%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	42%	100%	100%	23%	70%	88%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	(0.1, 0.95)-heterogeneous	in-arborescence	1%	2%	15%	0%	1%	10%
		ER, average degree 3	0%	2%	16%	0%	1%	8%
		ER, average degree 6	7%	10%	21%	2%	7%	14%
		SF, average degree 3	0%	6%	22%	0%	2%	14%
		SF, average degree 6	8%	19%	34%	4%	12%	21%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	2%	12%	0%	1%	11%
		ER, average degree 3	7%	14%	22%	4%	9%	18%
		ER, average degree 6	8%	18%	30%	6%	10%	20%
		SF, average degree 3	0%	9%	19%	0%	3%	17%
		SF, average degree 6	8%	12%	24%	4%	7%	18%
	homogeneous	in-arborescence	76%	78%	81%	32%	62%	81%
		ER, average degree 3	66%	100%	100%	26%	74%	100%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	29%	72%	100%	19%	53%	88%
		SF, average degree 6	100%	100%	100%	88%	100%	100%
$ V = 100$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	1%	12%	0%	1%	11%
		ER, average degree 3	0%	0%	15%	0%	1%	10%
		ER, average degree 6	6%	7%	16%	6%	6%	10%
		SF, average degree 3	0%	6%	23%	0%	0%	16%
		SF, average degree 6	8%	16%	30%	6%	12%	19%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	1%	11%	0%	1%	11%
		ER, average degree 3	6%	10%	18%	0%	7%	16%
		ER, average degree 6	7%	12%	22%	6%	8%	17%
		SF, average degree 3	0%	2%	18%	0%	0%	16%
		SF, average degree 6	5%	9%	18%	2%	6%	16%
	homogeneous	in-arborescence	76%	78%	81%	36%	67%	81%
		ER, average degree 3	76%	100%	100%	22%	73%	93%
		ER, average degree 6	100%	100%	100%	100%	100%	100%
		SF, average degree 3	26%	52%	100%	20%	42%	88%
		SF, average degree 6	100%	100%	100%	87%	100%	100%
$ V = 300$	(0.1, 0.95)-heterogeneous	in-arborescence	0%	1%	12%	0%	1%	12%
		ER, average degree 3	0%	2%	16%	0%	0%	5%
		ER, average degree 6	0%	6%	12%	0%	2%	6%
		SF, average degree 3	0%	9%	24%	0%	1%	16%
		SF, average degree 6	7%	16%	28%	6%	9%	19%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	1%	11%	0%	1%	11%
		ER, average degree 3	6%	8%	16%	0%	2%	16%
		ER, average degree 6	6%	8%	17%	6%	7%	16%
		SF, average degree 3	0%	1%	16%	0%	0%	16%
		SF, average degree 6	2%	6%	16%	2%	5%	16%

TABLE XII. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		idiosyncratic shock						
		$\Phi = 0.6, \gamma = 0.55$			$\Phi = 0.6, \gamma = 0.50$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	94%	95%	95%	53%	87%	95%
		ER, average degree 3	100%	100%	100%	62%	95%	100%
		ER, average degree 6	57%	58%	60%	57%	57%	57%
		SF, average degree 3	54%	100%	100%	45%	90%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	(0.1, 0.95)-heterogeneous	in-arborescence	2%	4%	16%	1%	3%	11%
		ER, average degree 3	0%	1%	14%	0%	0%	5%
		ER, average degree 6	8%	11%	23%	5%	7%	12%
		SF, average degree 3	0%	7%	24%	0%	2%	11%
		SF, average degree 6	9%	21%	39%	7%	15%	25%
	(0.2, 0.6)-heterogeneous	in-arborescence	1%	3%	13%	1%	1%	11%
		ER, average degree 3	7%	14%	25%	5%	10%	19%
		ER, average degree 6	11%	21%	34%	7%	11%	23%
		SF, average degree 3	1%	11%	21%	0%	9%	18%
		SF, average degree 6	10%	15%	27%	7%	9%	19%
	homogeneous	in-arborescence	94%	94%	95%	69%	92%	95%
		ER, average degree 3	77%	100%	100%	45%	87%	100%
		ER, average degree 6	57%	57%	57%	57%	57%	57%
		SF, average degree 3	36%	81%	100%	34%	75%	100%
		SF, average degree 6	100%	100%	100%	95%	100%	100%
$ V = 100$	(0.1, 0.95)-heterogeneous	in-arborescence	1%	2%	13%	1%	1%	11%
		ER, average degree 3	1%	2%	17%	0%	1%	13%
		ER, average degree 6	7%	7%	18%	7%	7%	15%
		SF, average degree 3	0%	9%	25%	0%	3%	17%
		SF, average degree 6	11%	19%	32%	7%	12%	22%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	5%	9%	19%	1%	8%	17%
		ER, average degree 6	8%	13%	23%	7%	9%	19%
		SF, average degree 3	0%	3%	19%	0%	1%	17%
		SF, average degree 6	7%	9%	20%	7%	8%	17%
	homogeneous	in-arborescence	94%	95%	95%	93%	95%	96%
		ER, average degree 3	96%	100%	100%	48%	89%	100%
		ER, average degree 6	57%	57%	57%	57%	57%	57%
		SF, average degree 3	34%	70%	100%	31%	67%	100%
		SF, average degree 6	100%	100%	100%	93%	100%	100%
$ V = 300$	(0.1, 0.95)-heterogeneous	in-arborescence	1%	1%	12%	1%	1%	11%
		ER, average degree 3	0%	4%	16%	0%	1%	10%
		ER, average degree 6	7%	7%	16%	5%	7%	12%
		SF, average degree 3	0%	8%	25%	0%	3%	19%
		SF, average degree 6	9%	17%	29%	7%	11%	21%
	(0.2, 0.6)-heterogeneous	in-arborescence	0%	0%	9%	0%	0%	9%
		ER, average degree 3	7%	8%	17%	3%	7%	16%
		ER, average degree 6	7%	9%	18%	7%	7%	17%
		SF, average degree 3	0%	1%	17%	0%	0%	16%
		SF, average degree 6	7%	8%	17%	7%	7%	16%

TABLE XIII. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		idiosyncratic shock					
		$\Phi = 0.7, \gamma = 0.65$			$\Phi = 0.7, \gamma = 0.60$		
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$
homogeneous	in-arborescence	94%	95%	96%	73%	94%	95%
	ER, average degree 3	100%	100%	100%	81%	100%	100%
	ER, average degree 6	57%	59%	60%	57%	57%	57%
	SF, average degree 3	55%	100%	100%	55%	100%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$ (0.1, 0.95)-heterogeneous	in-arborescence	3%	7%	17%	1%	2%	12%
	ER, average degree 3	1%	5%	21%	0%	3%	13%
	ER, average degree 6	10%	18%	28%	7%	9%	19%
	SF, average degree 3	7%	9%	26%	0%	3%	19%
	SF, average degree 6	13%	24%	41%	9%	16%	27%
(0.2, 0.6)-heterogeneous	in-arborescence	1%	3%	14%	1%	1%	11%
	ER, average degree 3	7%	15%	27%	7%	10%	20%
	ER, average degree 6	13%	23%	35%	7%	13%	25%
	SF, average degree 3	1%	11%	23%	0%	6%	19%
	SF, average degree 6	11%	17%	29%	7%	9%	20%
homogeneous	in-arborescence	94%	95%	95%	79%	95%	95%
	ER, average degree 3	77%	100%	100%	57%	97%	100%
	ER, average degree 6	57%	57%	57%	57%	57%	57%
	SF, average degree 3	37%	78%	100%	35%	76%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$ (0.1, 0.95)-heterogeneous	in-arborescence	1%	3%	13%	1%	1%	12%
	ER, average degree 3	0%	4%	17%	0%	1%	14%
	ER, average degree 6	7%	9%	19%	7%	7%	16%
	SF, average degree 3	0%	10%	25%	0%	3%	19%
	SF, average degree 6	9%	18%	33%	8%	15%	25%
(0.2, 0.6)-heterogeneous	in-arborescence	1%	1%	11%	1%	1%	11%
	ER, average degree 3	6%	11%	21%	3%	9%	17%
	ER, average degree 6	7%	15%	25%	7%	9%	19%
	SF, average degree 3	0%	3%	19%	0%	1%	17%
	SF, average degree 6	7%	10%	21%	7%	8%	17%
homogeneous	in-arborescence	94%	95%	95%	85%	95%	95%
	ER, average degree 3	96%	100%	100%	67%	98%	100%
	ER, average degree 6	57%	57%	57%	57%	57%	57%
	SF, average degree 3	33%	71%	100%	33%	69%	100%
	SF, average degree 6	100%	100%	100%	99%	100%	100%
$ V = 300$ (0.1, 0.95)-heterogeneous	in-arborescence	1%	1%	12%	1%	1%	12%
	ER, average degree 3	1%	5%	16%	0%	2%	13%
	ER, average degree 6	7%	7%	16%	6%	7%	15%
	SF, average degree 3	0%	9%	25%	0%	3%	19%
	SF, average degree 6	7%	17%	31%	7%	13%	21%
(0.2, 0.6)-heterogeneous	in-arborescence	1%	1%	11%	1%	1%	11%
	ER, average degree 3	7%	8%	17%	3%	7%	17%
	ER, average degree 6	7%	9%	19%	7%	8%	17%
	SF, average degree 3	0%	3%	17%	0%	1%	16%
	SF, average degree 6	7%	8%	17%	7%	7%	16%

TABLE XIV. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		idiosyncratic shock						
		$\Phi = 0.8, \gamma = 0.75$			$\Phi = 0.8, \gamma = 0.70$			
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	
	homogeneous	in-arborescence	94%	95%	95%	83%	94%	95%
		ER, average degree 3	100%	100%	100%	89%	100%	100%
		ER, average degree 6	57%	60%	63%	57%	57%	58%
		SF, average degree 3	55%	100%	100%	52%	100%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$	(0.1, 0.95)-heterogeneous	in-arborescence	4%	9%	21%	1%	2%	12%
		ER, average degree 3	0%	7%	21%	0%	1%	15%
		ER, average degree 6	10%	21%	31%	7%	9%	17%
		SF, average degree 3	1%	11%	27%	0%	5%	20%
		SF, average degree 6	9%	23%	43%	9%	17%	31%
	(0.2, 0.6)-heterogeneous	in-arborescence	2%	5%	15%	1%	1%	12%
		ER, average degree 3	8%	18%	29%	7%	11%	21%
		ER, average degree 6	15%	25%	38%	7%	15%	27%
		SF, average degree 3	1%	12%	23%	0%	7%	19%
		SF, average degree 6	12%	21%	33%	7%	10%	22%
	homogeneous	in-arborescence	94%	95%	95%	84%	94%	95%
		ER, average degree 3	78%	100%	100%	68%	100%	100%
		ER, average degree 6	57%	57%	59%	57%	57%	57%
		SF, average degree 3	37%	79%	100%	35%	81%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$	(0.1, 0.95)-heterogeneous	in-arborescence	1%	3%	13%	1%	1%	13%
		ER, average degree 3	0%	4%	17%	0%	3%	15%
		ER, average degree 6	7%	9%	21%	7%	7%	16%
		SF, average degree 3	0%	9%	25%	0%	4%	19%
		SF, average degree 6	10%	19%	35%	7%	17%	25%
	(0.2, 0.6)-heterogeneous	in-arborescence	1%	2%	13%	1%	1%	11%
		ER, average degree 3	5%	12%	21%	5%	9%	19%
		ER, average degree 6	9%	16%	28%	7%	11%	20%
		SF, average degree 3	0%	4%	19%	0%	1%	17%
		SF, average degree 6	8%	11%	23%	7%	8%	17%
	homogeneous	in-arborescence	94%	95%	96%	92%	95%	95%
		ER, average degree 3	96%	100%	100%	73%	100%	100%
		ER, average degree 6	57%	57%	57%	57%	57%	57%
		SF, average degree 3	33%	67%	100%	31%	69%	100%
		SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 300$	(0.1, 0.95)-heterogeneous	in-arborescence	1%	1%	12%	1%	1%	12%
		ER, average degree 3	1%	5%	16%	1%	3%	16%
		ER, average degree 6	7%	7%	17%	7%	7%	15%
		SF, average degree 3	0%	9%	25%	0%	7%	22%
		SF, average degree 6	9%	17%	31%	7%	13%	25%
	(0.2, 0.6)-heterogeneous	in-arborescence	1%	1%	11%	1%	1%	11%
		ER, average degree 3	7%	8%	18%	6%	7%	17%
		ER, average degree 6	7%	9%	20%	7%	8%	17%
		SF, average degree 3	0%	2%	17%	0%	1%	17%
		SF, average degree 6	7%	8%	17%	7%	7%	17%

TABLE XV. Residual instabilities of homogeneous versus heterogeneous networks under coordinated shocks. The percentages shown are the percentages of networks for which $\xi < 0.05$ or $\xi < 0.1$ or $\xi < 0.2$.

		idiosyncratic shock					
		$\Phi = 0.9, \gamma = 0.85$			$\Phi = 0.9, \gamma = 0.80$		
		$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$	$\xi < 0.05$	$\xi < 0.1$	$\xi < 0.2$
homogeneous	in-arborescence	94%	95%	95%	93%	94%	95%
	ER, average degree 3	100%	100%	100%	96%	100%	100%
	ER, average degree 6	5%	5%	8%	57%	57%	59%
	SF, average degree 3	55%	100%	100%	53%	100%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 50$ (0.1, 0.95)-heterogeneous	in-arborescence	9%	11%	21%	1%	3%	14%
	ER, average degree 3	1%	9%	23%	0%	2%	14%
	ER, average degree 6	11%	21%	31%	7%	10%	22%
	SF, average degree 3	1%	11%	27%	0%	7%	20%
	SF, average degree 6	12%	23%	44%	9%	20%	35%
(0.2, 0.6)-heterogeneous	in-arborescence	2%	4%	15%	1%	2%	12%
	ER, average degree 3	8%	17%	31%	7%	12%	22%
	ER, average degree 6	17%	25%	39%	9%	17%	29%
	SF, average degree 3	1%	12%	25%	0%	8%	19%
	SF, average degree 6	17%	23%	35%	9%	13%	23%
homogeneous	in-arborescence	94%	95%	95%	92%	95%	95%
	ER, average degree 3	77%	100%	100%	74%	100%	100%
	ER, average degree 6	3%	3%	5%	57%	57%	57%
	SF, average degree 3	34%	81%	100%	37%	81%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 100$ (0.1, 0.95)-heterogeneous	in-arborescence	1%	3%	14%	1%	1%	12%
	ER, average degree 3	0%	6%	19%	0%	1%	16%
	ER, average degree 6	7%	13%	22%	7%	7%	17%
	SF, average degree 3	0%	9%	25%	0%	6%	19%
	SF, average degree 6	11%	21%	37%	7%	17%	29%
(0.2, 0.6)-heterogeneous	in-arborescence	1%	2%	13%	1%	1%	11%
	ER, average degree 3	6%	13%	23%	5%	9%	19%
	ER, average degree 6	9%	19%	31%	7%	11%	21%
	SF, average degree 3	0%	5%	21%	0%	2%	17%
	SF, average degree 6	9%	14%	25%	7%	8%	18%
homogeneous	in-arborescence	94%	95%	95%	94%	95%	95%
	ER, average degree 3	0%	0%	100%	85%	100%	100%
	ER, average degree 6	3%	3%	3%	57%	57%	57%
	SF, average degree 3	30%	69%	100%	31%	69%	100%
	SF, average degree 6	100%	100%	100%	100%	100%	100%
$ V = 300$ (0.1, 0.95)-heterogeneous	in-arborescence	1%	1%	12%	1%	1%	11%
	ER, average degree 3	0%	5%	17%	0%	3%	16%
	ER, average degree 6	7%	8%	17%	7%	7%	16%
	SF, average degree 3	0%	9%	25%	0%	7%	22%
	SF, average degree 6	10%	17%	33%	6%	15%	25%
(0.2, 0.6)-heterogeneous	in-arborescence	1%	1%	11%	1%	1%	11%
	ER, average degree 3	7%	9%	18%	6%	7%	17%
	ER, average degree 6	7%	10%	21%	7%	9%	17%
	SF, average degree 3	0%	1%	17%	0%	1%	17%
	SF, average degree 6	7%	9%	17%	7%	7%	16%

3.4.2 Effect of total external assets

A controversial belief regarding the cause of the collapse of many major financial institutions around 2007 asserts that removal of the separation between investment and consumer banking allowed a ripple effect of insolvencies of individual banks to other banks (41; 24). In the setting here, the quantity \mathcal{E}/\mathcal{J} controls the total (normalized) amount of external investments of all banks in the network. Thus, varying the ratio \mathcal{E}/\mathcal{J} allows us to investigate the role of the magnitude of total external investments on the stability of the banking network (see Table XVI). All the entries in Table XVI are close to 0, showing that heterogeneous networks exhibited *very small* average changes in the vulnerability index ξ when \mathcal{E}/\mathcal{J} was varied. Thus, it can be concluded as:

③ *for heterogeneous banking networks, global stabilities are affected very little by the amount of the total external asset \mathcal{E} in the system.*

Visual illustrations of ③ are shown in Figure 17 and Figure 18 for homogeneous and heterogeneous networks, respectively.

3.4.3 Effect of network connectivity

Although it is clear that connectivity properties of a banking network has a crucial effect on its stability, prior researchers have drawn mixed conclusions on this. For example, Allen and Gale (6) concluded that networks with *less* connectivity are *more* prone to contagion than networks with higher connectivity due to improved resilience of banking network topologies with higher connectivity via transfer of proportion of the losses in one bank's portfolio to more banks through interbank agreements. On the other hand, Babus (10) observed that, when the network connectivity is higher, liquidity can be redistributed in the system to make the risk of contagion lower, and Gai and Kapadia (50) observed that higher connectivity among banks leads to more contagion effect during a crisis. The mixed conclusions arise because links between banks have conceptually two *conflicting* effects on contagion, namely,

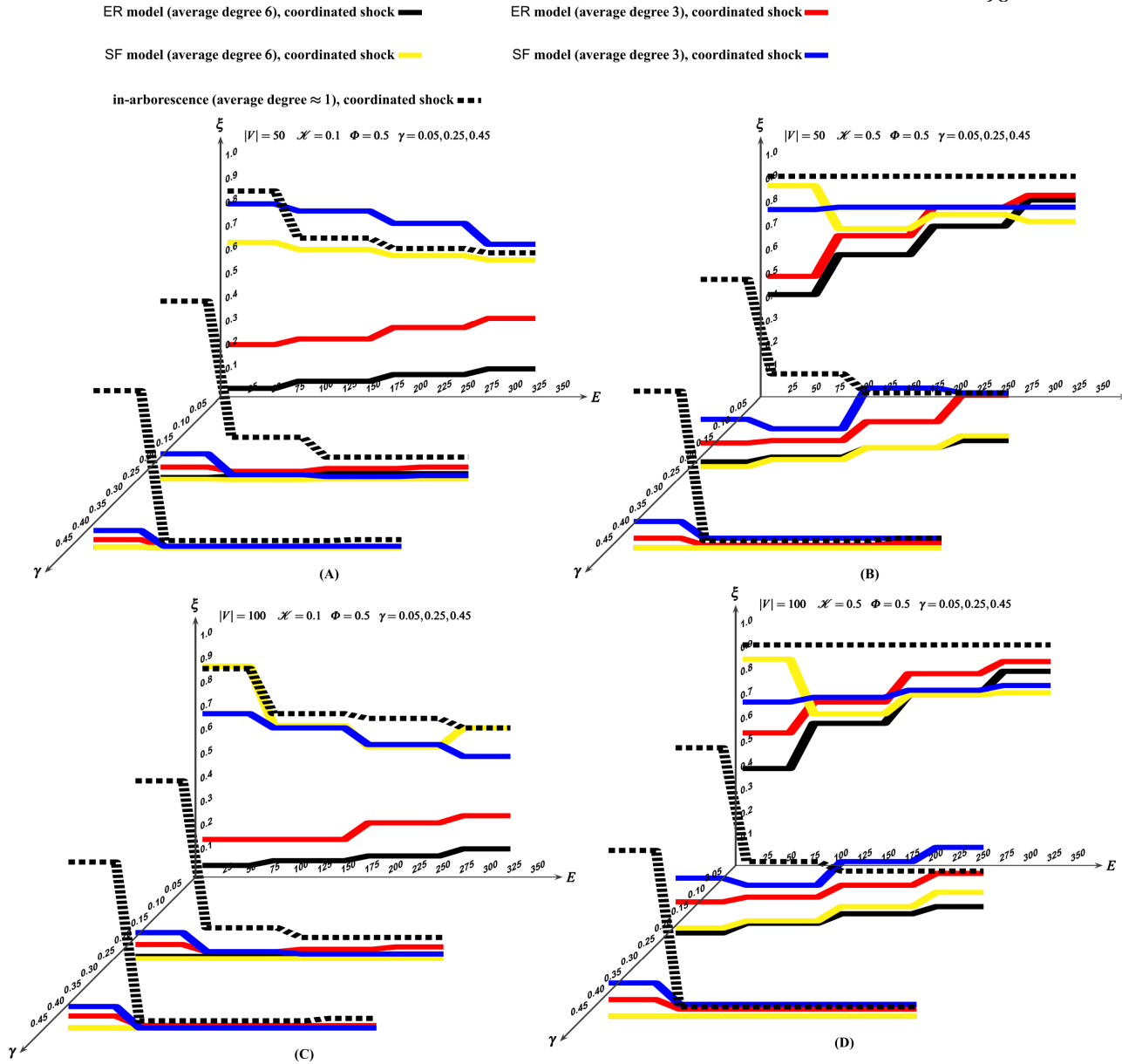


Figure 17. Effect of variations of the total external to internal asset ratio \mathcal{E}/\mathcal{I} on the vulnerability index ξ for homogeneous networks. Lower values of ξ imply higher global stability of a network.

- more interbank links increases the opportunity for spreading insolvencies to other banks,
- but, more interbank links also provide banks with co-insurance against fluctuating liquidity flows.

As the findings show, these two conflicting effects have different strengths in homogeneous versus highly heterogeneous networks, thus justifying the mixed conclusions of past researchers.

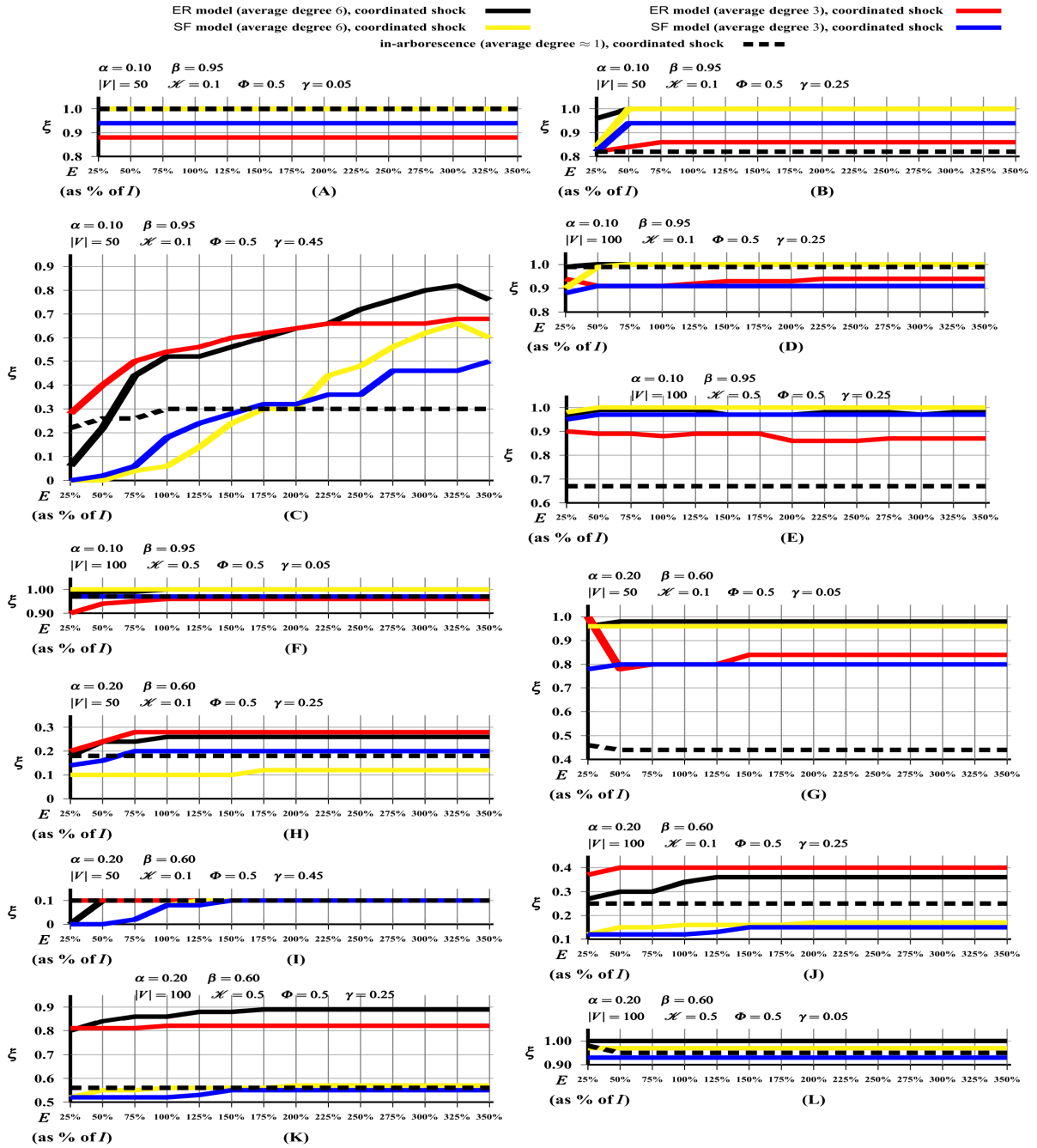


Figure 18. Effect of variations of the total external to internal asset ratio E/I on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

Homogeneous networks Recall that in a homogeneous network all banks have the same external asset.

Table XVII shows sparser homogeneous networks with lower average degrees to be more stable for same values of other parameters. Thus, it can be concluded as:

TABLE XVI. Absolute values of the largest change of the vulnerability index ξ in the range $0.25 \leq \mathcal{E}/\mathcal{J} \leq 3.5$.

	average values of $\left \max_{0.25 \leq \frac{\mathcal{E}}{\mathcal{J}} \leq 3.5} \{\xi\} - \min_{0.25 \leq \frac{\mathcal{E}}{\mathcal{J}} \leq 3.5} \{\xi\} \right $	
	coordinated shock	idiosyncratic shock
(0.1, 0.95)-heterogeneous in-arborescence	0.017	0.045
(0.2, 0.6)-heterogeneous in-arborescence	0.007	0.017
(0.1, 0.95)-heterogeneous ER, average degree 3	0.066	0.073
(0.2, 0.6)-heterogeneous ER, average degree 3	0.040	0.041
(0.1, 0.95)-heterogeneous ER, average degree 6	0.111	0.116
(0.2, 0.6)-heterogeneous ER, average degree 6	0.084	0.078
(0.1, 0.95)-heterogeneous SF, average degree 3	0.119	0.094
(0.2, 0.6)-heterogeneous SF, average degree 3	0.034	0.032
(0.1, 0.95)-heterogeneous SF, average degree 6	0.200	0.179
(0.2, 0.6)-heterogeneous SF, average degree 6	0.054	0.054

TABLE XVII. Effect of connectivity on the stability for homogeneous networks under coordinated and idiosyncratic shocks. The percentage shown for a comparison of the type “network A versus network B” indicates the percentage of data points for which the stability of networks of type A was at least as much as that of networks of type B.

ER average degree 3 versus ER average degree 6		SF average degree 3 versus SF average degree 6	
coordinated shock	idiosyncratic shock	coordinated shock	idiosyncratic shock
97.43%	97.05%	98.89%	98.29%

④ *for homogeneous networks, higher connectivity leads to lower stability.*

Heterogeneous networks In a heterogeneous network, there *are* banks that are “too big” in the sense that these banks have much larger external assets and inter-bank exposures compared to the remaining banks.

Table XVIII shows that for heterogeneous network models denser networks with higher average degree are

TABLE XVIII. Effect of connectivity on the stability under coordinated and idiosyncratic shocks for **(A)** (α, β) -heterogeneous ER and SF networks and **(B)** (α, β) -heterogeneous in-arborescence versus (α, β) -heterogeneous SF networks. The percentage shown for a comparison of the type “network A versus network B” indicates the percentage of data points for which the stability of networks of type A was at least as much as that of networks of type B.

(0,1,0.95) ER average degree 6 versus (0,1,0.95) ER average degree 3		(0,2,0.6) ER average degree 6 versus (0,2,0.6) ER average degree 3		(0,1,0.95) SF average degree 6 versus (0,1,0.95) SF average degree 3		(0,2,0.6) SF average degree 6 versus (0,2,0.6) SF average degree 3	
coordinated shock	idiosyncratic shock	coordinated shock	idiosyncratic shock	coordinated shock	idiosyncratic shock	coordinated shock	idiosyncratic shock
89.3%	82.39%	68.12%	61.46%	85.51%	73.81%	69.29%	73.07%

(A)

(0,1,0.95) SF average degree 3 and average degree 6 versus (0,1,0.95)-heterogeneous in-arborescence (SF ave. degree 1)		(0,2,0.6) SF average degree 3 and average degree 6 versus (0,2,0.6)-heterogeneous in-arborescence (SF ave. degree 1)	
coordinated shock	idiosyncratic shock	coordinated shock	idiosyncratic shock
85.7%	81.86%	56.21%	51.07%

(B)

more stable than the corresponding sparser networks for same values of other parameters, especially when the heterogeneity of the network is larger (*i.e.*, when $\alpha = 0.1, \beta = 0.95$). Thus, it can be concluded as:

⑤ *for heterogeneous networks, higher connectivity leads to higher stability.*

Formal intuition behind the conclusions in ④ and ⑤

Informally, conclusions ④ and ⑤ indicate that in homogeneous networks higher connectivity leads to more opportunity for spreading insolvencies to other banks whereas in heterogeneous networks higher connectivity provides banks with co-insurance against fluctuating liquidity flows through shared interbank assets. However, a precise formal treatment of mechanism that drives such conclusions is complicated due to several reasons such as the random nature of the networks, the randomness in asset distribution for heterogeneous

networks and the non-linear nature of the insolvency propagation equation. Nevertheless, The following, somewhat simplified, formal reasoning is provided. The following notations and conventions are used¹:

- $\text{deg}_{\text{ave}} = \frac{|E|}{n}$ will denote the average degree of a graph G . It is assumed that deg_{ave} is a small positive integer constant independent of n (e.g., in our simulation work, $\text{deg}_{\text{ave}} \in \{1, 3, 6\}$).
- Δx will denote a small change for the value x of a variable.
- For two functions $f(r)$ and $g(r)$ of a variable r , the notation $f \approx g$ will be used (respectively, $f \lesssim g$, $f \gtrsim g$) if $\lim_{r \rightarrow \infty} \frac{f(r)}{g(r)} = 1$ (respectively, $\lim_{r \rightarrow \infty} \frac{f(r)}{g(r)} \leq 1$, $\lim_{r \rightarrow \infty} \frac{f(r)}{g(r)} \geq 1$).
- The standard phrase “with high probability” (or **w.h.p.** in short) refers to a probability $p(n)$ such that $\lim_{n \rightarrow \infty} p(n) = 0$.
- The superscripts “homo” and “hetero” will be used if necessary to denote the value of a quantity for homogeneous and heterogeneous networks, respectively.

Consider a node $v \in V_{\text{shock}}$ with $\text{deg}_{\text{in}}(v) > 1$ and suppose that v fails due to the initial shock at $t = 0$.

By Equation (Equation 2.1), for every edge $(u, v) \in E$, the amount of shock u receives from v is given by

$\mathbb{B} = \min \{\mathbb{A}, c_1\}$ with

$$\begin{aligned} \mathbb{A} &= \frac{\Phi(c_1 \text{deg}_{\text{in}}(v) - c_1 \text{deg}_{\text{out}}(v) + c_2 \mathcal{E}) - \gamma(c_1 \text{deg}_{\text{in}}(v) + c_2 \mathcal{E})}{\text{deg}_{\text{in}}(v)} \\ &= c_1(\Phi - \gamma) + c_2(\Phi - \gamma) \frac{\mathcal{E}}{\text{deg}_{\text{in}}(v)} - c_1 \Phi \frac{\text{deg}_{\text{out}}(v)}{\text{deg}_{\text{in}}(v)} \end{aligned} \quad (3.1)$$

for some appropriate positive quantities c_1 and c_2 that may be estimated as follows:

- If G is homogeneous then $c_1^{\text{homo}} = \frac{\mathcal{I}}{n \text{deg}_{\text{ave}}} = 1$ and $c_2^{\text{homo}} = \frac{1}{n}$.

¹In standard algorithmic analysis terminologies, $f \approx g$ implies $\frac{f(r)}{g(r)} = 1 \pm o(1)$.

- If G is (α, β) -heterogeneous then c_1^{hetero} and c_2^{hetero} are random variables independent of deg_{ave} . Using the notations in Definition 3.3.1 the expected value of c_2^{hetero} may be estimated as follows:

$$\mathbb{E}[c_2^{\text{hetero}}] = \frac{\Pr[v \in \widetilde{V}] \frac{\beta \mathcal{E}}{\alpha n} + \Pr[v \notin \widetilde{V}] \frac{(1-\beta) \mathcal{E}}{(1-\alpha)n}}{\mathcal{E}} = \frac{\alpha \frac{\beta \mathcal{E}}{\alpha n} + (1-\alpha) \frac{(1-\beta) \mathcal{E}}{(1-\alpha)n}}{\mathcal{E}} = \frac{1}{n}$$

The expected value of c_1^{hetero} depend on the nature (SF or ER) of the random network; its estimation is therefore deferred until later.

The goal here is to provide evidence for a claim of the following nature for either random SF or random ER networks:

The case of random SF networks

If G is a directed SF network, then the discrete probability density function for the degree of any node v in G is given by:

$$\forall k \in \{1, 2, \dots, n-1\}: \Pr[\text{deg}_{\text{in}}(v) = k] = \Pr[\text{deg}_{\text{out}}(v) = k] = C k^{-\mu}$$

where $\mu > 2$ is the constant for the exponent of the distribution and $C > 0$ is a constant such that $\mathbb{E}[\text{deg}_{\text{in}}(v)] = \mathbb{E}[\text{deg}_{\text{out}}(v)] = \text{deg}_{\text{ave}}$. For example, for the random in-arborescence networks, the results in (12) imply $\mu = 3$. To simplify exposure, in the following it is assumed that $\mu = 3$, though the analysis can be extended in a straightforward manner for any other $\mu > 2$. $\zeta(s) = \sum_{x=1}^{\infty} x^{-s}$ is the well-known Riemann zeta function; it is well known that $\zeta(s) \approx \sum_{x=1}^{n-1} x^{-s}$ for any $s > 2$ and for all large n (see (65, page 74-75)) and $\zeta(s) < \frac{2^{s-1}}{2^{s-1}-1}$ for any $s > 1$ (see (65, page 489)). In particular, it is known that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.2020569 \dots$ and $\zeta(4) = \frac{\pi^4}{90}$. Note that

$$\mathbb{E}[\text{deg}_{\text{in}}(v)] = \text{deg}_{\text{ave}} \equiv \sum_{k=1}^{n-1} k (Ck^{-3}) = \text{deg}_{\text{ave}} \Rightarrow C \approx \frac{\text{deg}_{\text{ave}}}{\zeta(2)} = \frac{6 \text{ deg}_{\text{ave}}}{\pi^2}$$

Lemma 3.4.2. $\mathbb{E} \left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0 \right] \approx \frac{\pi^2}{15} \deg_{\text{ave}}$ and $\text{Var} [\deg_{\text{in}}(v)] \approx \frac{6 \deg_{\text{ave}}}{\pi^2} \ln n$.

Proof of Lemma 3.4.2

Using standard probabilistic calculations, we get

$$\mathbb{E} \left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0 \right] = \sum_{k=1}^{n-1} \frac{1}{k} \Pr [\deg_{\text{in}}(v) = k] = C \sum_{k=1}^{n-1} k^{-4} \approx C \zeta(4) \approx \frac{\pi^2}{15} \deg_{\text{ave}}$$

$$\begin{aligned} \text{Var} [\deg_{\text{in}}(v)] &= \mathbb{E} [(\deg_{\text{in}}(v))^2] - (\mathbb{E} [\deg_{\text{in}}(v)])^2 = \sum_{k=1}^{n-1} k^2 (Ck^{-3}) - (\deg_{\text{ave}})^2 \\ &= C \sum_{k=1}^{n-1} \frac{1}{k} - (\deg_{\text{ave}})^2 \approx \frac{6 \deg_{\text{ave}}}{\pi^2} \ln n - (\deg_{\text{ave}})^2 \approx \frac{6 \deg_{\text{ave}}}{\pi^2} \ln n \end{aligned}$$

An estimation of c_1^{hetero} using the notations in Definition 3.3.1 is provided here.

Lemma 3.4.3. *W.h.p.* $1 + \alpha - \beta - \frac{\alpha\beta}{2} \leq \mathbb{E} [c_1^{\text{hetero}}] \leq \frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha}$.

Proof of Lemma 3.4.3

Let $\mathcal{D} = \sum_{v \in \bar{V}} \deg_{\text{in}}(v) + \sum_{v \in \bar{V}} \deg_{\text{out}}(v)$. By linearity of expectation, we have

$$\mathbb{E} [\mathcal{D}] = 2 \sum_{v \in \bar{V}} \mathbb{E} [\deg_{\text{in}}(v)] = 2 \alpha n \deg_{\text{ave}}$$

and similarly, since $\deg_{\text{in}}(v)$ is independent of any other $\deg_{\text{in}}(u)$ for $u \neq v$, we have

$$\text{Var} [\mathcal{D}] = 2 \sum_{v \in \bar{V}} \text{Var} [\deg_{\text{in}}(v)] \approx \frac{12 \deg_{\text{ave}}}{\pi^2} \alpha n \ln n$$

Thus, via Chebyshev's inequality (7, page 37), for any positive λ we have

$$\Pr \left[|\mathcal{D} - \mathbb{E} [\mathcal{D}]| \geq \lambda \sqrt{\text{Var} [\mathcal{D}]} \right] \leq \frac{1}{\lambda^2} \equiv \Pr \left[|\mathcal{D} - 2 \alpha n \deg_{\text{ave}}| \geq \lambda \sqrt{\frac{12 \deg_{\text{ave}}}{\pi^2} \alpha n \ln n} \right] \leq \frac{1}{\lambda^2}$$

Setting $\lambda = \sqrt{\frac{\pi^2 \ln n}{12\alpha}}$ gives $\Pr\left[|\mathcal{D} - 2\alpha n \deg_{\text{ave}}| \geq \sqrt{n \deg_{\text{ave}} \ln n}\right] \leq \frac{12\alpha}{\pi^2 \ln n}$ and thus w.h.p. $\mathcal{D} \approx 2\alpha n \deg_{\text{ave}}$. Since $\frac{\mathcal{D}}{2} \leq |\tilde{E}| \leq \mathcal{D}$, it now follows that

$$\begin{aligned} \alpha n \deg_{\text{ave}} &\leq \mathbb{E}[|\tilde{E}|] \leq 2\alpha n \deg_{\text{ave}} \\ \text{w.h.p. } \alpha n \deg_{\text{ave}} &\leq |\tilde{E}| \leq 2\alpha n \deg_{\text{ave}} \end{aligned}$$

Also, note that $\Pr[(u, v) \in \tilde{E}] = \Pr[v \in \tilde{V}] = \alpha$. For notational convenience, let \tilde{E}_1 be a random subset of $\alpha|\tilde{E}|$ of edges from the edges in \tilde{E} as used in Definition 3.3.1. This implies that

$$\begin{aligned} \mathbb{E}[c_1^{\text{hetero}}] &= \Pr[(u, v) \in \tilde{E}_1] \left(\frac{\beta \mathcal{I}}{\alpha |\tilde{E}|} \right) + \left(1 - \Pr[(u, v) \in \tilde{E}_1] \right) \frac{\mathcal{I} - \beta \mathcal{I}}{|E| - \alpha |\tilde{E}|} \\ &= \alpha \Pr[(u, v) \in \tilde{E}] \left(\frac{\beta \mathcal{I}}{\alpha |\tilde{E}|} \right) + \left(1 - \alpha \Pr[(u, v) \in \tilde{E}] \right) \frac{\mathcal{I} - \beta \mathcal{I}}{|E| - \alpha |\tilde{E}|} \\ &= \alpha^2 \left(\frac{\beta \mathcal{I}}{\alpha |\tilde{E}|} \right) + (1 - \alpha^2) \frac{\mathcal{I} - \beta \mathcal{I}}{|E| - \alpha |\tilde{E}|} \\ \Rightarrow \text{w.h.p. } \alpha^2 \frac{\beta n \deg_{\text{ave}}}{2\alpha n \deg_{\text{ave}}} + (1 - \alpha^2) \frac{(1 - \beta) n \deg_{\text{ave}}}{n \deg_{\text{ave}} - \alpha n \deg_{\text{ave}}} \\ &\leq \mathbb{E}[c_1^{\text{hetero}}] \leq \alpha^2 \frac{\beta n \deg_{\text{ave}}}{\alpha n \deg_{\text{ave}}} + (1 - \alpha^2) \frac{(1 - \beta) n \deg_{\text{ave}}}{n \deg_{\text{ave}} - 2\alpha n \deg_{\text{ave}}} \\ &\equiv \text{w.h.p. } \frac{\alpha\beta}{2} + (1 + \alpha)(1 - \beta) \leq \mathbb{E}[c_1^{\text{hetero}}] \leq \alpha\beta + \frac{(1 - \alpha^2)(1 - \beta)}{1 - 2\alpha} \\ &\equiv \text{w.h.p. } 1 + \alpha - \beta - \frac{\alpha\beta}{2} \leq \mathbb{E}[c_1^{\text{hetero}}] \leq \frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \end{aligned}$$

Due to the above lemma, it can be assumed that

$$\begin{aligned} \text{w.h.p. } 0.156875 &\leq \mathbb{E}[c_1^{\text{hetero}}] \leq 0.16, \quad \text{if } G \text{ is } (0.1, 0.95)\text{-heterogeneous} \\ 0.54 &\leq \mathbb{E}[c_1^{\text{hetero}}] \leq 0.76, \quad \text{if } G \text{ is } (0.2, 0.6)\text{-heterogeneous} \end{aligned}$$

The sensitivity of the amount of shock \mathbb{A} transmitted from v to u as the average degree \deg_{ave} is changed while keeping all other parameters the same will be investigated. For notational convenience, the parameters

are normalized such that $\mathcal{I} = n \deg_{\text{ave}}$ at the initial value d of \deg_{ave} . As \deg_{ave} is increased from d to $d + \Delta d$, \mathcal{I} is still kept the same. Thus,

$$\mathcal{I} \Big|_{\deg_{\text{ave}}=d+\Delta d}^{\text{at}} = \mathcal{I} \Big|_{\deg_{\text{ave}}=d}^{\text{at}} \Rightarrow c_1^{\text{homo}} \Big|_{\deg_{\text{ave}}=d+\Delta d}^{\text{at}} = \frac{\mathcal{I} \Big|_{\deg_{\text{ave}}=d+\Delta d}^{\text{at}}}{n(d+\Delta d)} = \frac{nd}{n(d+\Delta d)} = \frac{d}{d+\Delta d}$$

Equation (Equation 3.1) gives the following for homogeneous and heterogeneous networks:

- If G is a homogeneous network then

$$\begin{aligned} & \mathbb{E}[\mathbb{A}^{\text{homo}} \mid \deg_{\text{ave}} = d] \\ & \approx (\Phi - \gamma) + \frac{\mathcal{E}(\Phi - \gamma)}{n} \mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0\right] - \Phi \mathbb{E}[\deg_{\text{out}}(v)] \mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0\right] \\ & = (\Phi - \gamma) + \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} d - \frac{\pi^2 \Phi}{15} d^2 \end{aligned}$$

$$\begin{aligned} & \mathbb{E}[\mathbb{A}^{\text{homo}} \mid \deg_{\text{ave}} = d + \Delta d] \\ & \approx \frac{d}{d+\Delta d} (\Phi - \gamma) + \frac{\mathcal{E}(\Phi - \gamma)}{n} \mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0\right] - \frac{d}{d+\Delta d} \Phi \mathbb{E}[\deg_{\text{out}}(v)] \mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{in}}(v) > 0\right] \\ & = \frac{d}{d+\Delta d} (\Phi - \gamma) + \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} (d + \Delta d) - \frac{d}{d+\Delta d} \frac{\pi^2 \Phi}{15} (d + \Delta d)^2 \end{aligned}$$

$$\begin{aligned} & \Delta \mathbb{E}[\mathbb{A}^{\text{homo}}] \\ & = \mathbb{E}[\mathbb{A}^{\text{homo}} \mid \deg_{\text{ave}} = d + \Delta d] - \mathbb{E}[\mathbb{A}^{\text{homo}} \mid \deg_{\text{ave}} = d] \\ & \approx -\frac{\Delta d}{d+\Delta d} (\Phi - \gamma) + \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} \Delta d - \frac{\pi^2 \Phi}{15} d \Delta d \end{aligned}$$

- If G is a heterogeneous network then

$$\begin{aligned} \mathbb{E}[A^{\text{hetero}}] &= (\Phi - \gamma) \mathbb{E}[c_1^{\text{hetero}}] + \frac{\mathcal{E}(\Phi - \gamma)}{n} \mathbb{E}\left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0\right] \\ &\quad - \Phi \mathbb{E}[c_1^{\text{hetero}}] \mathbb{E}[\text{deg}_{\text{out}}(v)] \mathbb{E}\left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0\right] \end{aligned}$$

which provides the following bounds:

$$\mathbb{E}[A^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d] \approx (\Phi - \gamma) \mathbb{E}[c_1^{\text{hetero}}] + \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} d - \frac{\pi^2 \Phi}{15} d^2 \mathbb{E}[c_1^{\text{hetero}}]$$

$$\mathbb{E}[A^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d + \Delta d] \approx (\Phi - \gamma) \mathbb{E}[c_1^{\text{hetero}}] + \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} (d + \Delta d) - \frac{\pi^2 \Phi}{15} d^2 \mathbb{E}[c_1^{\text{hetero}}]$$

$$\begin{aligned} \Delta \mathbb{E}[A^{\text{hetero}}] &= \mathbb{E}[A^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d + \Delta d] - \mathbb{E}[A^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d] \\ &\approx \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} \Delta d - \frac{\pi^2 \Phi}{15} d \Delta d \mathbb{E}[c_1^{\text{hetero}}] \end{aligned}$$

Now note that

$$\begin{aligned} \Delta \mathbb{E}[A^{\text{homo}}] \lesssim 0 &\Rightarrow \frac{\Phi - \gamma}{d + \Delta d} + \frac{\pi^2 \Phi d}{15} > \frac{\pi^2 \mathcal{E}(\Phi - \gamma)}{15n} \\ &\equiv \left(\frac{1}{d + \Delta d} + \frac{\pi^2 d}{15} - \frac{\pi^2 \mathcal{E}}{15n} \right) \Phi > \left(\frac{1}{d + \Delta d} - \frac{\pi^2 \mathcal{E}}{15n} \right) \gamma \quad (3.2) \end{aligned}$$

$$\begin{aligned} \Delta \mathbb{E}[A^{\text{hetero}}] \gtrsim 0 &\Rightarrow \frac{\mathcal{E}(\Phi - \gamma)}{n} > \Phi d \mathbb{E}[c_1^{\text{hetero}}] \equiv \left(\frac{\mathcal{E}}{n} - d \mathbb{E}[c_1^{\text{hetero}}] \right) \Phi > \left(\frac{\mathcal{E}}{n} \right) \gamma \\ &\Rightarrow \text{w.h.p.} \left(\frac{\mathcal{E}}{n} - d \frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \right) \Phi > \left(\frac{\mathcal{E}}{n} \right) \gamma \quad (3.3) \end{aligned}$$

It is easy to verify that constraints (Equation 3.2) and (Equation 3.3) are satisfied for many natural combinations of parameters. In fact, the constraints (Equation 3.2) and (Equation 3.3) are almost always satisfied when \mathcal{E} grows moderately linearly with n . To see this informally, note that since $\alpha \ll \beta$ and α is small, $\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \approx 1 - \beta$ and thus (Equation 3.3) is approximately

$$\left(\frac{\mathcal{E}}{n} - d(1 - \beta)\right) \Phi > \left(\frac{\mathcal{E}}{n}\right) \gamma \quad (\text{Equation 3.3}')$$

Now suppose that $\mathcal{E} < d(1 - \beta)n$. Then, (Equation 3.2) and ((Equation 3.3')) are always satisfied since $\Phi > \gamma$. For a numerical example, suppose that G is a (0.1, 0.95)-heterogeneous network (i.e., $\alpha = 0.1$ and $\beta = 0.95$), $d = 3$, $\Delta d = 1$, $\gamma = 0.2$ and $\Phi = 0.4$. Then constraints (Equation 3.2) and (Equation 3.3) reduce to:

$$0.5 + 0.4\pi^2 - \frac{2\pi^2\mathcal{E}}{15n} > 0.25 - \frac{\pi^2\mathcal{E}}{15n} \quad \text{and} \quad \frac{2\mathcal{E}}{n} - 0.94125 > \frac{\mathcal{E}}{n}$$

and these constraints can be satisfied when \mathcal{E} grows moderately linearly with n , i.e., when $0.94125n < \mathcal{E} < 6.38n$.

The case of random ER networks

In a random ER network, the probability of having a particular edge is given by the following set of independent Bernoulli trials:

$$\forall u, v \in V \text{ with } u \neq v : \Pr[(u, v) \in E] = p = \frac{\text{deg}_{\text{ave}}}{n - 1}$$

Thus, for every $k \in \{0, 1, 2, \dots, n - 1\}$:

$$\Pr[\text{deg}_{\text{in}}(v) = k] = \Pr[\text{deg}_{\text{out}}(v) = k] = \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$$

$$\mathbb{E}[\text{deg}_{\text{in}}] = \mathbb{E}[\text{deg}_{\text{out}}] = p(n-1) = \text{deg}_{\text{ave}}$$

TABLE XIX. Values of $\Upsilon(x)$ and $\frac{\partial \Upsilon(x)}{\partial x}$ for a few small integral values of x using straightforward calculations.

	$x =$										
	2	3	4	5	6	7	8	9	10	11	12
$\Upsilon(x) \approx$	0.499	0.411	0.324	0.256	0.207	0.172	0.147	0.128	0.113	0.101	0.091
$\frac{\partial \Upsilon(x)}{x} \approx$	0.432	0.284	0.245	0.256	0.199	0.166	0.143	0.125	0.111	0.100	0.090

$$\text{Var}[\text{deg}_{\text{in}}(v)] = \text{Var}[\text{deg}_{\text{out}}(v)] = p(1-p)(n-1) = \left(1 - \frac{\text{deg}_{\text{ave}}}{n-1}\right) \text{deg}_{\text{ave}}$$

Since $\text{deg}_{\text{ave}} = p(n-1)$ is a constant, one can approximate the above binomial distribution by a Poisson's distribution (83, page 72) to obtain

$$\Pr[\text{deg}_{\text{in}}(v) = k] = \Pr[\text{deg}_{\text{out}}(v) = k] \approx e^{-\text{deg}_{\text{ave}}} \frac{(\text{deg}_{\text{ave}})^k}{k!}$$

Lemma 3.4.4. $\left| \mathbb{E}\left[\frac{1}{\text{deg}_{\text{in}}(v)}\right] - \sum_{k=1}^{\lfloor 3 \text{deg}_{\text{ave}} + 10 \rfloor} \frac{1}{k} e^{-\text{deg}_{\text{ave}}} \frac{(\text{deg}_{\text{ave}})^k}{k!} \right| \lesssim 10^{-10}$ and $\frac{\partial}{\partial d} \mathbb{E}\left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{ave}} = d\right] \approx \frac{1-e^{-d}}{d}$.

For notational convenience, let $\Upsilon(x) = \sum_{k=1}^{\lfloor 3x+10 \rfloor} \frac{1}{k} e^{-x} \frac{x^k}{k!}$. Since 10^{-10} is an extremely small number, $\mathbb{E}\left[\frac{1}{\text{deg}_{\text{in}}(v)}\right] \approx \Upsilon(\text{deg}_{\text{ave}})$ will be used in the sequel. Values of $\Upsilon(x)$ and $\frac{\partial \Upsilon(x)}{\partial x} = \frac{1-e^{-x}}{x}$ for a few small integral values of x are shown in Table XIX. It is easy to see that $\lim_{x \rightarrow \infty} \Upsilon(x) = 0$ and, for large x , $\frac{\partial \Upsilon(x)}{\partial x} \approx 1/x$.

Proof of Lemma 3.4.4

$\mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)}\right] = \sum_{k=1}^{n-1} \frac{1}{k} \Pr[\deg_{\text{in}}(v) = k] = \sum_{k=1}^{n-1} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!}$. It is easy to see that extending the finite series to an infinite series does not change the asymptotic value of the series since

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} - \sum_{k=1}^{n-1} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} \\
&= \sum_{k=n}^{\infty} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} \\
&\leq \frac{e^{-\deg_{\text{ave}}}}{n} \sum_{k=n}^{\infty} \frac{(\deg_{\text{ave}})^k}{k!} \\
&\leq \left(\frac{e^{-\deg_{\text{ave}}}}{n}\right) \left(\frac{(\deg_{\text{ave}})^n}{n!}\right) \left(\max_{0 \leq x \leq \deg_{\text{ave}}} \{e^x\}\right) && \text{using the Lagrange remainder term} \\
&= \left(\frac{e^{-\deg_{\text{ave}}}}{n}\right) \left(\frac{(\deg_{\text{ave}})^n}{n!}\right) e^{\deg_{\text{ave}}} && \text{for the Maclaurin series expansion of } e^{\deg_{\text{ave}}} \\
&= \frac{(\deg_{\text{ave}})^n}{n(n!)}
\end{aligned}$$

and $\lim_{n \rightarrow \infty} \frac{(\deg_{\text{ave}})^n}{n(n!)}$ because \deg_{ave} is a constant independent of n . Thus, it can be concluded that $\sum_{k=1}^{\infty} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} \approx$

$\sum_{k=1}^{n-1} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!}$. It now follows that

$$\begin{aligned}
\frac{\partial}{\partial d} \mathbb{E}\left[\frac{1}{\deg_{\text{in}}(v)} \mid \deg_{\text{ave}} = d\right] &\approx \frac{\partial}{\partial d} \sum_{k=1}^{\infty} \frac{1}{k} e^{-d} \frac{d^k}{k!} \\
&= e^{-d} \sum_{k=1}^{\infty} \frac{d^{k-1}}{k!} = \frac{e^{-d}}{d} \sum_{k=1}^{\infty} \frac{d^k}{k!} = \frac{e^{-d}}{d} (e^d - 1) = \frac{1 - e^{-d}}{d}
\end{aligned}$$

This proves one of the claims in the lemma. To prove the other claim, using a well-known approximation on the first inverse moment of Poisson's distribution (59, page 173) we have

$$\left| \sum_{k=1}^{\infty} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} - \sum_{k=1}^{\lfloor 3 \deg_{\text{ave}} + 10 \rfloor} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} \right| < 10^{-10}$$

and therefore we obtain

$$\left| \sum_{k=1}^{n-1} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} - \sum_{k=1}^{\lfloor 3 \deg_{\text{ave}} + 10 \rfloor} \frac{1}{k} e^{-\deg_{\text{ave}}} \frac{(\deg_{\text{ave}})^k}{k!} \right| \lesssim 10^{-10}$$

As before, an estimation of c_1^{hetero} using the notations in Definition 3.3.1 is provided.

Lemma 3.4.5. *W.h.p.* $1 + \alpha - \beta - \frac{\alpha\beta}{2} \leq \mathbb{E}[c_1^{\text{hetero}}] \leq \frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha}$.

Proof of Lemma 3.4.5

Let $\mathcal{D} = \sum_{v \in \tilde{V}} \deg_{\text{in}}(v) + \sum_{v \in \tilde{V}} \deg_{\text{out}}(v)$. The proof of Lemma 3.4.3 can be reused provided that $\mathbb{E}[\mathcal{D}] = 2\alpha n \deg_{\text{ave}}$ and w.h.p. $\mathcal{D} \approx 2\alpha n \deg_{\text{ave}}$. By linearity of expectation, we have $\mathbb{E}[\mathcal{D}] = 2 \sum_{v \in \tilde{V}} \mathbb{E}[\deg_{\text{in}}(v)] = 2\alpha n \deg_{\text{ave}}$, and similarly $\text{Var}[\mathcal{D}] = 2 \sum_{v \in \tilde{V}} \text{Var}[\deg_{\text{in}}(v)] = 2\alpha n \left(1 - \frac{\deg_{\text{ave}}}{n-1}\right) \deg_{\text{ave}} \approx 2\alpha n \deg_{\text{ave}}$. Thus, via Chebyshev's inequality, for any positive λ we have

$$\Pr\left[|\mathcal{D} - \mathbb{E}[\mathcal{D}]| \geq \lambda \sqrt{\text{Var}[\mathcal{D}]}\right] \leq \frac{1}{\lambda^2} \equiv \Pr\left[|\mathcal{D} - 2\alpha n \deg_{\text{ave}}| \gtrsim \lambda \sqrt{2\alpha n \deg_{\text{ave}}}\right] \leq \frac{1}{\lambda^2}$$

Setting $\lambda = \sqrt{\frac{\ln n}{2\alpha}}$ gives

$$\Pr\left[|\mathcal{D} - 2\alpha n \deg_{\text{ave}}| \gtrsim \sqrt{n \deg_{\text{ave}} \ln n}\right] \leq \frac{2\alpha}{\ln n}$$

and thus w.h.p. $\mathcal{D} \approx 2\alpha n \deg_{\text{ave}}$.

Using the above result, the following bounds hold:

- If G is a homogeneous network then

$$\frac{\partial}{\partial d} c_1^{\text{homo}} \Big|_{\deg_{\text{ave}}=d}^{\text{at}} = \lim_{\Delta d \rightarrow 0} \frac{\frac{\mathcal{I} \Big|_{\deg_{\text{ave}}=d+\Delta d}^{\text{at}}}{n(d+\Delta d)} - \frac{\mathcal{I} \Big|_{\deg_{\text{ave}}=d}^{\text{at}}}{nd}}{\Delta d} = \lim_{\Delta d \rightarrow 0} \frac{\frac{nd}{n(d+\Delta d)} - 1}{\Delta d} = -\frac{1}{d}$$

$$\begin{aligned}
& \mathbb{E} \left[\mathbb{A}^{\text{homo}} \mid \text{deg}_{\text{ave}} = d \right] \\
&= c_1^{\text{homo}} (\Phi - \gamma) + \frac{\mathcal{E}(\Phi - \gamma)}{n} \mathbb{E} \left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0 \right] - c_1^{\text{homo}} \Phi \mathbb{E}[\text{deg}_{\text{out}}(v)] \mathbb{E} \left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0 \right] \\
&\approx c_1^{\text{homo}} (\Phi - \gamma) + \frac{\mathcal{E}(\Phi - \gamma)}{n} \Upsilon(d) - c_1^{\text{homo}} \Phi d \Upsilon(d)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial d} \mathbb{E} \left[\mathbb{A}^{\text{homo}} \mid \text{deg}_{\text{ave}} = d \right] \\
&\approx (\Phi - \gamma) \frac{\partial}{\partial d} c_1^{\text{homo}} \Big|_{\text{deg}_{\text{ave}}=d} + \frac{\mathcal{E}(\Phi - \gamma)}{n} \frac{\partial}{\partial d} \Upsilon(d) - \Phi \frac{\partial}{\partial d} \left(c_1^{\text{homo}} \Big|_{\text{deg}_{\text{ave}}=d} d \Upsilon(d) \right) \\
&= -\frac{\Phi - \gamma}{d} + \left(\frac{\mathcal{E}(\Phi - \gamma)}{n} \right) \left(\frac{1 - e^{-d}}{d} \right) - \Phi \left(-\Upsilon(d) + \Upsilon(d) + 1 - e^{-d} \right) \quad \text{using product rule} \\
& \quad \text{of derivatives} \\
&= -\frac{\Phi - \gamma}{d} + \left(\frac{\mathcal{E}(\Phi - \gamma)}{n} \right) \left(\frac{1 - e^{-d}}{d} \right) - \Phi (1 - e^{-d})
\end{aligned}$$

- If G is a heterogeneous network then

$$\begin{aligned}
\mathbb{E}[\mathbb{A}^{\text{hetero}}] &= (\Phi - \gamma) \mathbb{E} \left[c_1^{\text{hetero}} \right] + \frac{\mathcal{E}(\Phi - \gamma)}{n} \mathbb{E} \left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0 \right] \\
& \quad - \Phi \mathbb{E} \left[c_1^{\text{hetero}} \right] \mathbb{E}[\text{deg}_{\text{out}}(v)] \mathbb{E} \left[\frac{1}{\text{deg}_{\text{in}}(v)} \mid \text{deg}_{\text{in}}(v) > 0 \right]
\end{aligned}$$

which provides the following bounds w.h.p.:

$$\mathbb{E} \left[\mathbb{A}^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d \right] \approx (\Phi - \gamma) \mathbb{E} \left[c_1^{\text{hetero}} \right] + \frac{\mathcal{E}(\Phi - \gamma)}{n} \Upsilon(d) - \Phi d \Upsilon(d) \mathbb{E} \left[c_1^{\text{hetero}} \right]$$

$$\begin{aligned}
& \frac{\partial}{\partial d} \mathbb{E} \left[\mathbb{A}^{\text{hetero}} \mid \text{deg}_{\text{ave}} = d \right] \\
&\approx \frac{\mathcal{E}(\Phi - \gamma)}{n} \frac{\partial}{\partial d} \Upsilon(d) - \Phi \left(\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \right) \frac{\partial}{\partial d} (d \Upsilon(d))
\end{aligned}$$

$$= \left(\frac{\mathcal{E}(\Phi - \gamma)}{n} \right) \left(\frac{1 - e^{-d}}{d} \right) - \Phi \left(\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \right) (1 - e^{-d} + \Upsilon(d))$$

using product
rule of
derivatives

Now note the following:

$$\begin{aligned} \frac{\partial}{\partial d} \mathbb{E} [A^{\text{homo}} | \text{deg}_{\text{ave}} = d] &\lesssim 0 \\ &\Rightarrow \frac{\Phi - \gamma}{d} + \Phi (1 - e^{-d}) > \left(\frac{\mathcal{E}(\Phi - \gamma)}{n} \right) \left(\frac{1 - e^{-d}}{d} \right) \\ &\equiv \left(\frac{1}{d} + 1 - e^{-d} - \frac{\mathcal{E}}{n} \left(\frac{1 - e^{-d}}{d} \right) \right) \Phi > \left(\frac{1}{d} - \frac{\mathcal{E}}{n} \left(\frac{1 - e^{-d}}{d} \right) \right) \gamma \quad (3.4) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial d} \mathbb{E} [A^{\text{hetero}} | \text{deg}_{\text{ave}} = d] &\gtrsim 0 \\ &\Rightarrow \left(\frac{\mathcal{E}(\Phi - \gamma)}{n} \right) \left(\frac{1 - e^{-d}}{d} \right) > \Phi \left(\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \right) (1 - e^{-d} + \Upsilon(d)) \\ &\equiv \left(\frac{\mathcal{E}}{n} \left(\frac{1 - e^{-d}}{d} \right) - \left(\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \right) (1 - e^{-d} + \Upsilon(d)) \right) \Phi > \frac{\mathcal{E}}{n} \left(\frac{1 - e^{-d}}{d} \right) \gamma \quad (3.5) \end{aligned}$$

It is easy to verify that constraints (Equation 3.4) and (Equation 3.5) are satisfied for many natural combinations of parameters. For example, the constraints (Equation 3.4) and (Equation 3.5) are almost always satisfied if d is sufficiently large. To see this informally, note that if d is large then $\frac{1}{d}, e^{-d}, \Upsilon(d) \approx 0$. Moreover, since $\alpha \ll \beta$ and α is small, $\frac{1 + \alpha\beta - \alpha^2 - \beta - \alpha^2\beta}{1 - 2\alpha} \approx 1 - \beta$ and then constraints (Equation 3.4) and (Equation 3.5) reduce to

$$\left(1 - \frac{\mathcal{E}}{nd} \right) \Phi \gtrsim - \frac{\mathcal{E}}{nd} \gamma \equiv \Phi \gtrsim \frac{\frac{\mathcal{E}}{nd}}{\frac{\mathcal{E}}{nd} - 1} \gamma \quad (\text{Equation 3.4})^{d \gg 0}$$

TABLE XX. Comparisons of strengths of coordinated versus idiosyncratic shocks. The percentages indicate the percentage of total number of data points (combinations of parameters Φ , γ , \mathcal{E} and \mathcal{K}) for that network type that resulted in $\xi_c \geq \xi_r$, where ξ_c and ξ_r are the vulnerability indices under coordinated and idiosyncratic shocks, respectively.

(α, β) -heterogeneous networks									
In-arborescence		ER average degree 3		ER average degree 6		SF average degree 3		SF average degree 6	
$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$
$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$	$\beta = 0.95$	$\beta = 0.6$
56.64%	57.27%	89.66%	90.97%	98.99%	98.04%	93.16%	64.13%	94.44%	66.48%

homogeneous networks				
In-arborescence	ER average degree 3	ER average degree 6	SF average degree 3	SF average degree 6
84.62%	74.97%	78.59%	81.15%	54.80%

$$\left(\frac{\mathcal{E}}{nd} - 1 + \beta \right) \Phi \gtrsim \frac{\mathcal{E}}{nd} \gamma \equiv \Phi \gtrsim \frac{\frac{\mathcal{E}}{nd}}{\frac{\mathcal{E}}{nd} - 1 + \beta} \gamma \quad (\text{Equation 3.5})^{d \gg 0}$$

If $\mathcal{E} > nd$ then $\frac{\mathcal{E}}{nd} > 1$ and $\frac{\mathcal{E}}{\frac{\mathcal{E}}{nd} - 1 + \beta} < \frac{\mathcal{E}}{\frac{\mathcal{E}}{nd} - 1}$; thus both (Equation 3.4) ^{$d \gg 0$} and (Equation 3.5) ^{$d \gg 0$} can be satisfied by choosing Φ appropriately with respect to γ . For a numerical example, suppose that G is a (0.1, 0.95)-heterogeneous network (i.e., $\alpha = 0.1$ and $\beta = 0.95$), $d = 10$, and $\mathcal{E} = 12n$. Then constraints (Equation 3.4),(Equation 3.5) reduce to:

$$1.17 < \frac{\Phi}{\gamma} < 11$$

which corresponds to most settings of Φ and γ used in the simulation.

3.4.3.1 Random versus correlated initial failures

For most parameter combinations, the results, tabulated in Table XX, show that coordinated shocks, which are a type of correlated shocks, resulted in insolvencies of higher number of nodes as opposed to

idiosyncratic shocks for the same network with the same parameters, *often by a factor of two or more*. For example, Table XX shows that for (0.1, 0.95)-heterogeneous ER networks of average degree 6 the vulnerability index under coordinated shocks is at least as much as the vulnerability index under idiosyncratic shocks 98.99% of the time. Thus, it can be concluded that:

⑥ *correlated shocking mechanisms are more appropriate to measure the worst-case stability compared to idiosyncratic shocking mechanisms.*

For visual illustrations of ⑥, see Figure 20—Figure 24.

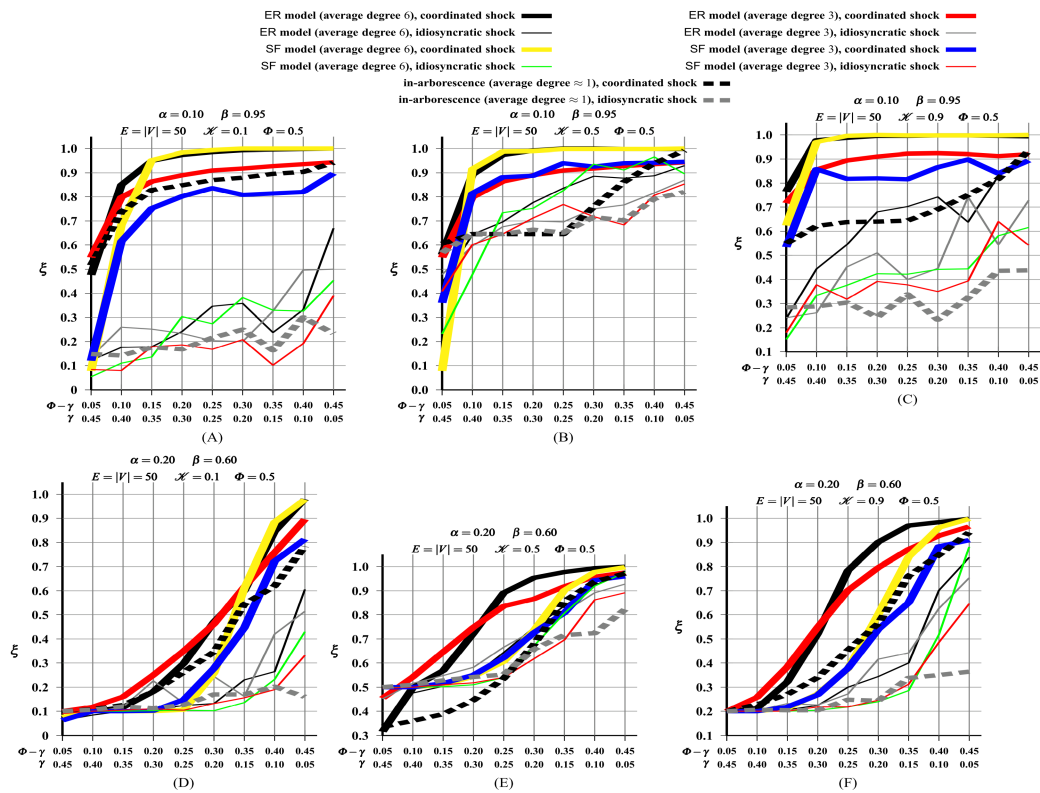


Figure 19. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

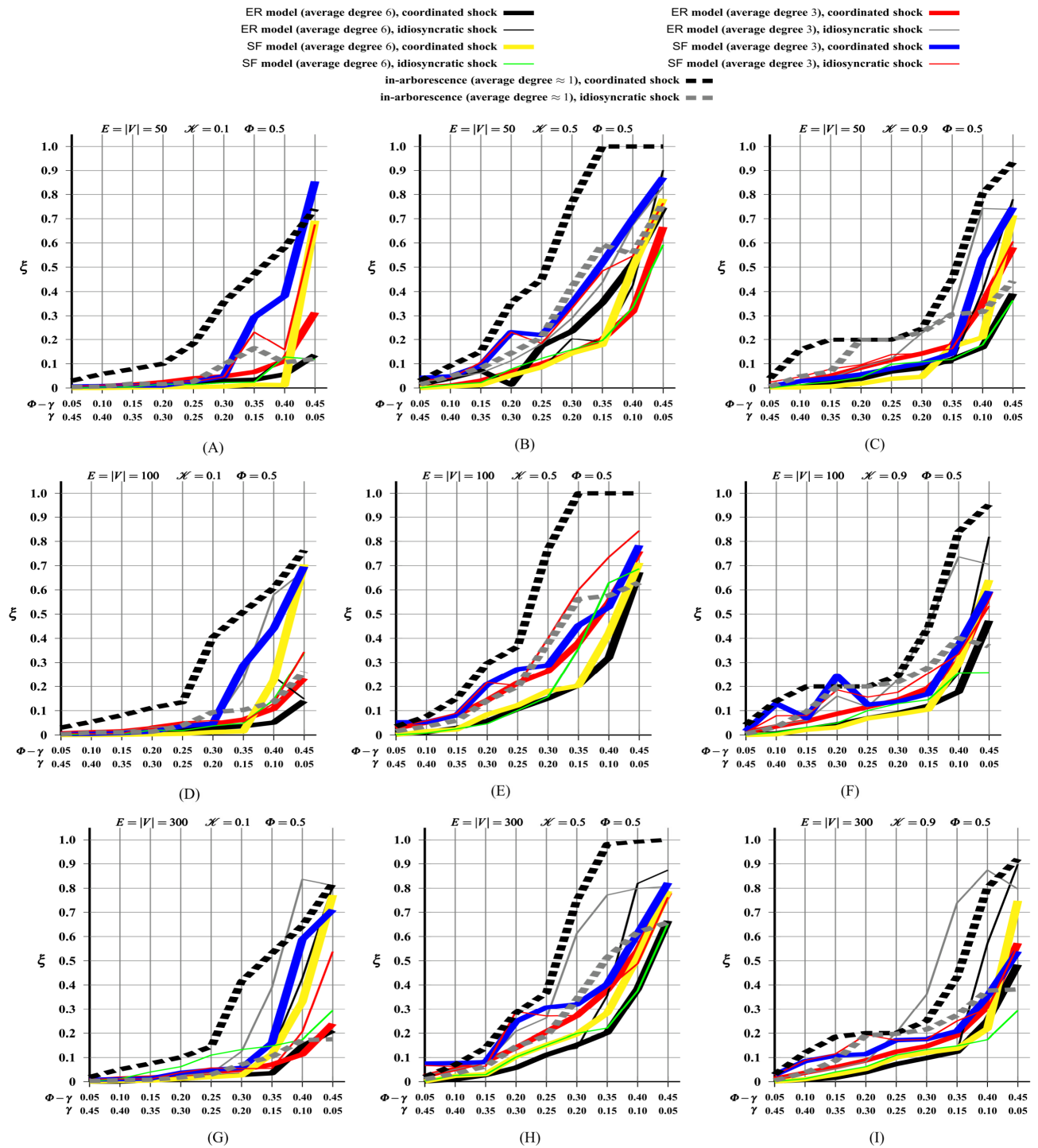


Figure 20. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for homogeneous networks. Lower values of ξ imply higher global stability of a network.

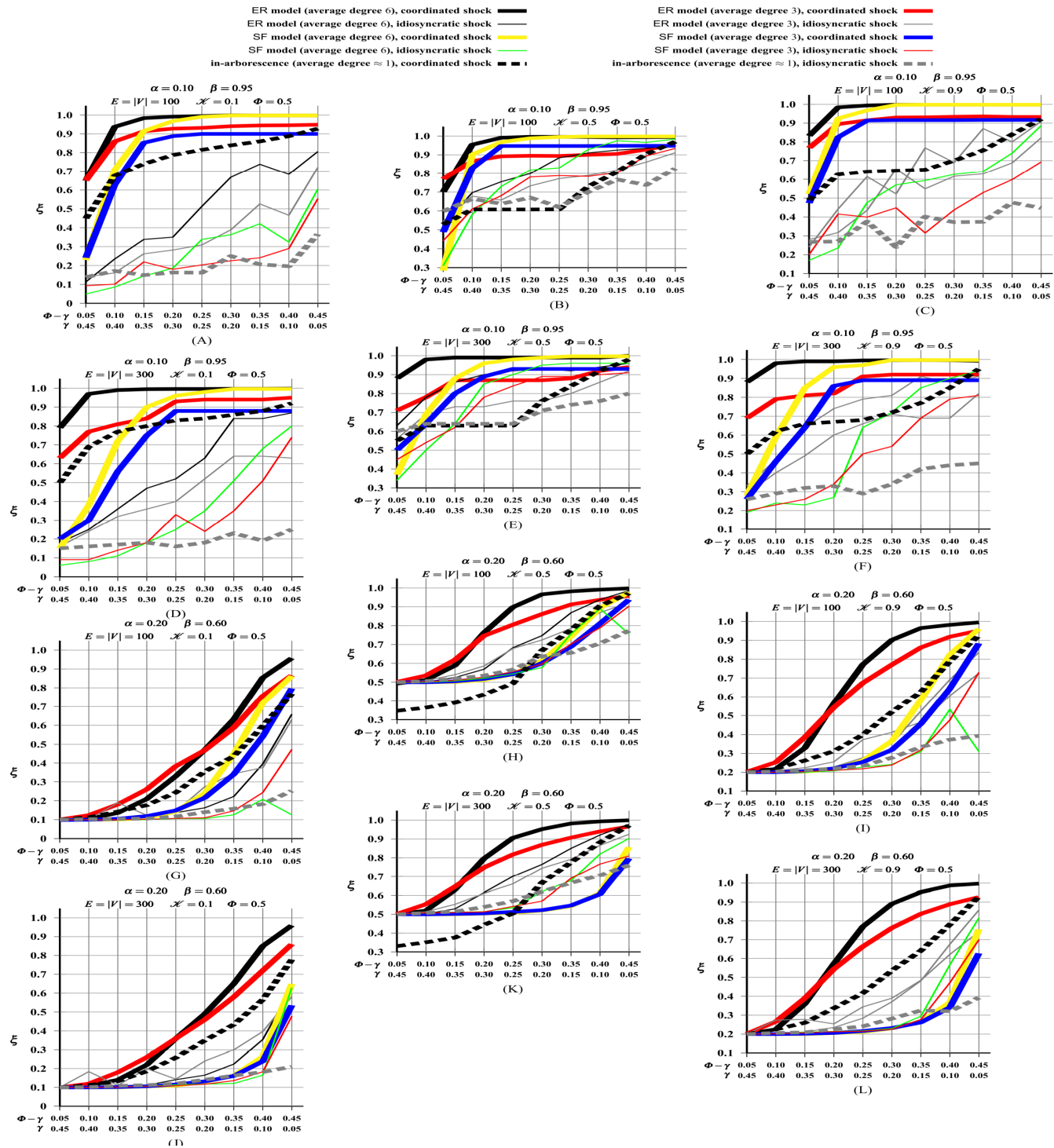


Figure 21. Effect of variations of equity to asset ratio (with respect to shock) on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

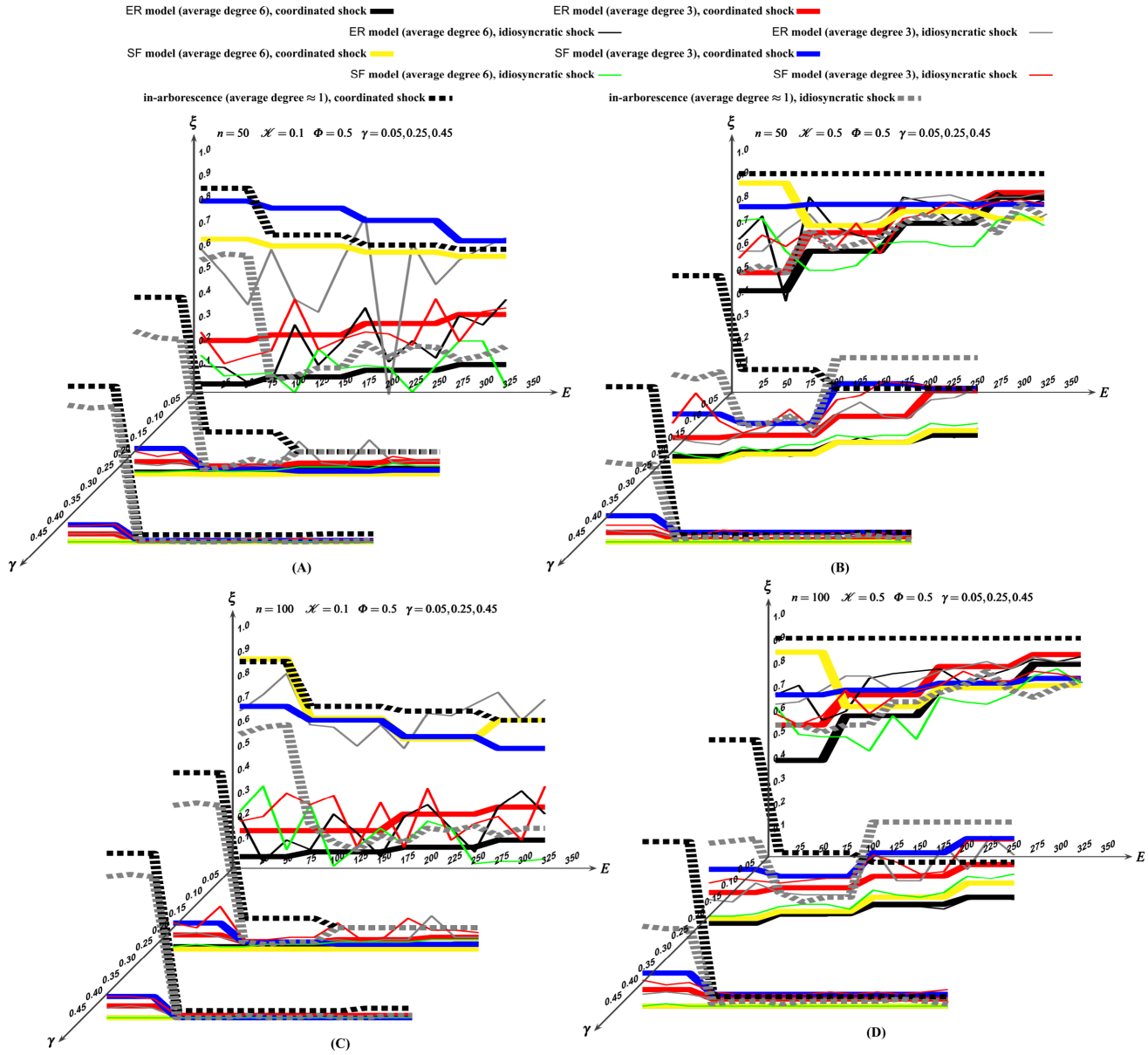


Figure 22. Effect of variations of the total external to internal asset ratio E/I on the vulnerability index ξ for homogeneous networks. Lower values of ξ imply higher global stability of a network.

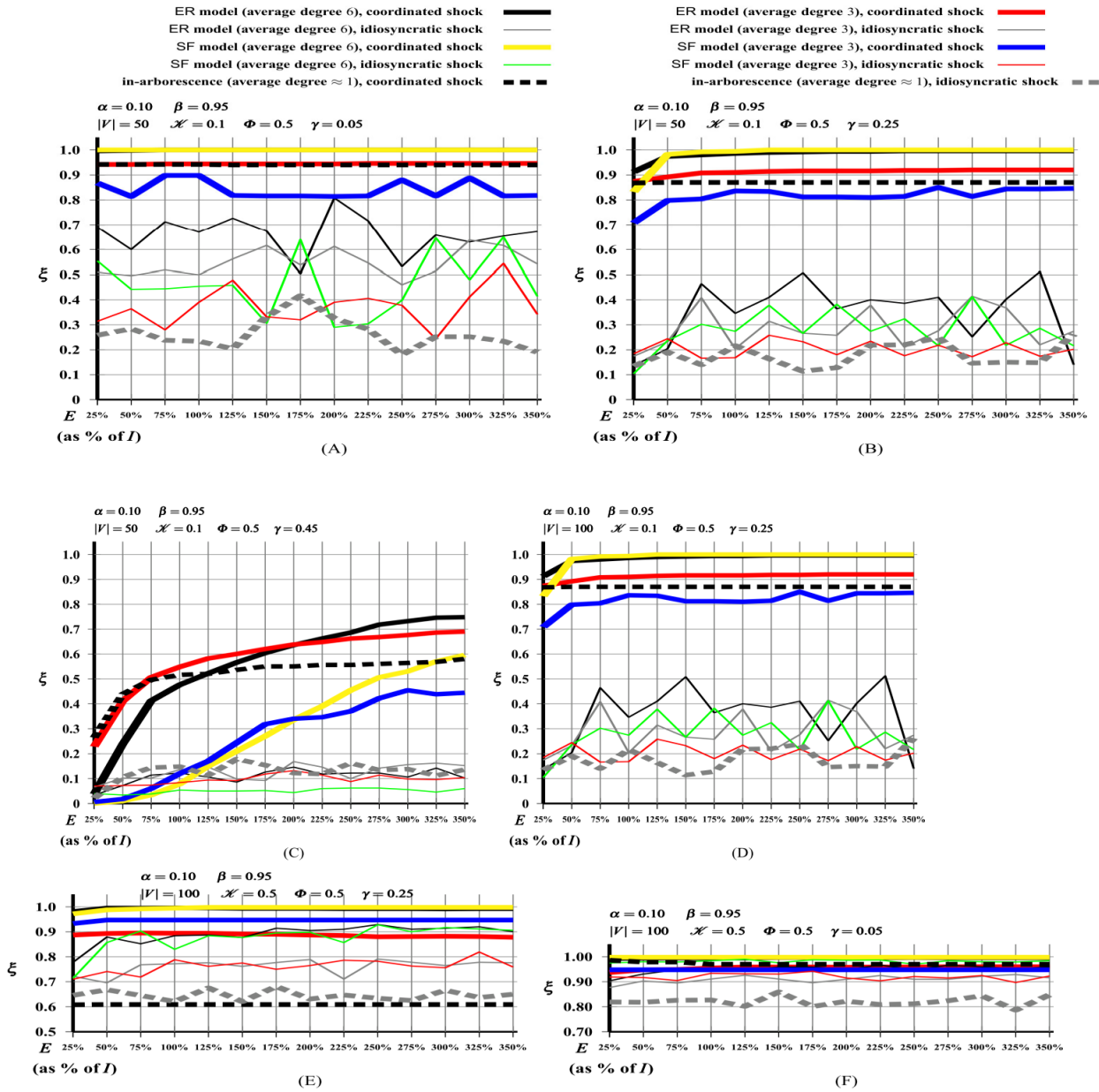


Figure 23. Effect of variations of the total external to internal asset ratio \mathcal{E}/\mathcal{I} on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

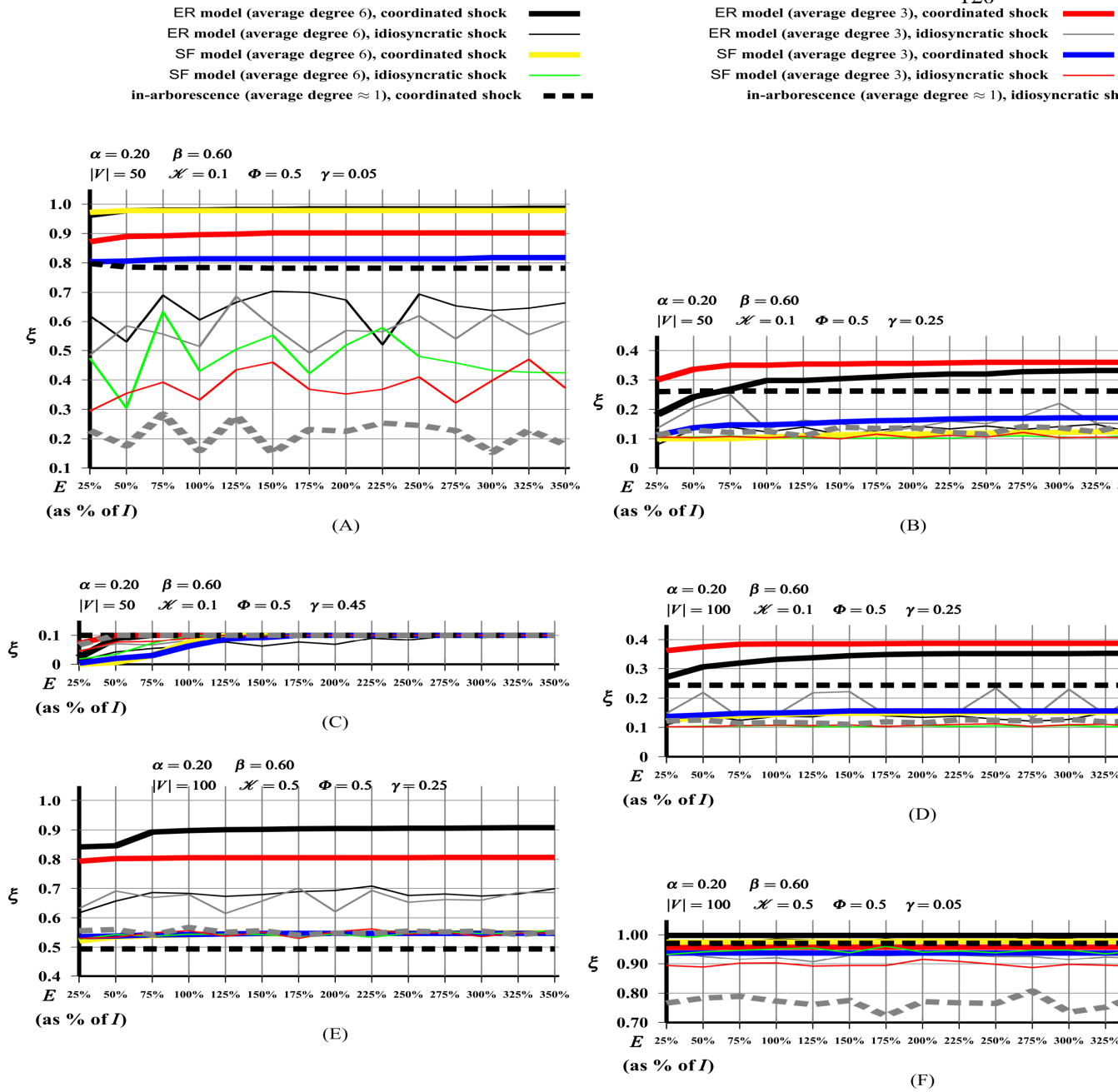


Figure 24. Effect of variations of the total external to internal asset ratio $\frac{E}{I}$ on the vulnerability index ξ for (α, β) -heterogeneous networks. Lower values of ξ imply higher global stability of a network.

3.4.3.2 Phase transition properties of stability

Phase transitions are quite common when one studies various topological properties of graphs (19). The stability measure exhibits several sharp *phase transitions* for various banking networks and combinations of parameters; see Figure 14—Figure 17 for visual illustrations. Two such interesting phase transitions are discussed in the following, with an intuitive theoretical explanation for one of them.

3.4.3.3 Dense homogeneous networks

Based on the behavior of ξ with respect to $(\Phi - \gamma)$, it was observed that, for smaller value of \mathcal{K} and for denser ER and SF networks under either coordinated or idiosyncratic shocks, there is often a *sharp* decrease of stability when γ was decreased beyond a particular threshold value. For example, with $\Phi = 0.5$ and $\mathcal{K} = 0.1$, 100-node dense (average degree 6) SF and ER homogeneous networks exhibited more than ninefold increase in ξ around $\gamma = 0.15$ and $\gamma = 0.1$, respectively; see Figure 14 for visual illustrations.

To investigate the global extent of such a sharp decrease at a threshold value of γ in the range $[0.05, 0.2]$, for each of the five homogeneous network types under coordinated shocks and for each values of the parameters $|V|$, Φ , $\frac{\mathcal{E}}{\mathcal{J}}$ and \mathcal{K} is computed, the ratio

$$\Lambda\left(n, \Phi, \frac{\mathcal{E}}{\mathcal{J}}, \mathcal{K}\right) = \frac{\max_{0.05 \leq \gamma \leq 0.2} \{\xi\} - \min_{0.05 \leq \gamma \leq 0.2} \{\xi\}}{\max_{\text{entire range of } \gamma} \{\xi\} - \min_{\text{entire range of } \gamma} \{\xi\}}$$

that provides the *maximum* percentage of the total change of the vulnerability index that occurred within this narrow range of γ . The values of $\Lambda\left(n, \Phi, \frac{\mathcal{E}}{\mathcal{J}}, \mathcal{K}\right)$ for the dense ER and SF homogeneous networks under coordinated shocks are shown in Table XXI for $\Phi = 0.5, 0.8$ and $\mathcal{K} = 0.1, 0.2, 0.3, 0.4, 0.5$ (the behaviour of ξ is similar for other intermediate values of Φ). If the growth of ξ with respect to γ was uniform or near uniform over the entire range of γ , Λ would be approximately $\lambda = \frac{0.2-0.05}{0.45-0.05} = 0.375$; thus, any value of Λ significantly higher than λ indicates a sharp transition within the above-mentioned range of values of γ . As Table XXI shows, a *significant majority* of the entries for $\Phi \leq 0.8$ and $\kappa \leq 0.2$ are 2λ or more.

TABLE XXI. Values of $\Lambda(n, \Phi, E/I, \mathcal{K}) = \frac{\max_{0.05 \leq \gamma \leq 0.2} \{\xi\} - \min_{0.05 \leq \gamma \leq 0.2} \{\xi\}}{\max_{\text{entire range of } \gamma} \{\xi\} - \min_{\text{entire range of } \gamma} \{\xi\}}$ for homogeneous dense ER and SF networks under coordinated shocks. Entries that are at least 2×0.375 are shown in **boldface**.

E/I [V]	$\Phi = 0.5$										$\Phi = 0.8$									
	ER average degree 6					SF average degree 6					ER average degree 6					SF average degree 6				
	$\mathcal{K} =$					$\mathcal{K} =$					$\mathcal{K} =$					$\mathcal{K} =$				
	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
50	0.79	0.77	0.79	0.83	0.81	0.99	0.98	0.97	0.95	0.93	0.81	0.77	0.77	0.74	0.7	0.95	0.9	0.85	0.78	0.72
0.25 100	0.8	0.79	0.81	0.77	0.77	0.99	0.97	0.95	0.92	0.9	0.84	0.81	0.79	0.76	0.72	0.95	0.86	0.79	0.71	0.66
300	0.93	0.89	0.86	0.83	0.81	0.98	0.96	0.93	0.9	0.88	0.83	0.8	0.79	0.76	0.71	0.89	0.79	0.71	0.63	0.59
50	0.79	0.77	0.79	0.83	0.81	0.99	0.98	0.97	0.95	0.93	0.83	0.78	0.78	0.74	0.68	0.95	0.89	0.84	0.78	0.72
0.5 100	0.8	0.79	0.81	0.77	0.77	0.99	0.97	0.95	0.92	0.9	0.83	0.81	0.79	0.76	0.71	0.94	0.85	0.78	0.71	0.65
300	0.93	0.89	0.86	0.83	0.81	0.98	0.96	0.93	0.9	0.88	0.83	0.81	0.79	0.76	0.7	0.88	0.78	0.7	0.63	0.61
50	0.79	0.77	0.79	0.83	0.81	0.99	0.98	0.97	0.95	0.93	0.84	0.78	0.77	0.73	0.67	0.95	0.89	0.83	0.77	0.71
0.75 100	0.8	0.79	0.81	0.77	0.77	0.99	0.97	0.95	0.92	0.9	0.83	0.82	0.8	0.76	0.71	0.94	0.85	0.77	0.7	0.65
300	0.93	0.89	0.86	0.83	0.81	0.98	0.96	0.93	0.9	0.88	0.83	0.8	0.78	0.74	0.69	0.9	0.77	0.69	0.63	0.61
50	0.78	0.77	0.78	0.79	0.79	0.99	0.96	0.93	0.88	0.82	0.83	0.79	0.78	0.72	0.67	0.95	0.89	0.83	0.76	0.69
1 100	0.8	0.8	0.8	0.8	0.78	0.98	0.93	0.87	0.81	0.75	0.83	0.81	0.8	0.76	0.7	0.95	0.83	0.74	0.66	0.61
300	0.87	0.82	0.83	0.81	0.77	0.97	0.91	0.87	0.81	0.75	0.82	0.79	0.78	0.74	0.69	0.86	0.71	0.61	0.57	0.55
50	0.78	0.77	0.78	0.79	0.79	0.99	0.96	0.93	0.88	0.82	0.84	0.79	0.79	0.73	0.67	0.95	0.89	0.83	0.76	0.69
1.25 100	0.8	0.8	0.8	0.8	0.78	0.98	0.93	0.87	0.81	0.75	0.83	0.82	0.8	0.76	0.7	0.95	0.83	0.74	0.67	0.61
300	0.87	0.82	0.83	0.81	0.77	0.97	0.91	0.87	0.81	0.75	0.83	0.8	0.78	0.74	0.68	0.86	0.7	0.63	0.59	0.56
50	0.78	0.77	0.78	0.79	0.79	0.99	0.96	0.93	0.88	0.82	0.84	0.8	0.79	0.72	0.65	0.94	0.89	0.83	0.76	0.69
1.5 100	0.8	0.8	0.8	0.8	0.78	0.98	0.93	0.87	0.81	0.75	0.83	0.81	0.79	0.74	0.68	0.94	0.81	0.72	0.67	0.62
300	0.87	0.82	0.83	0.81	0.77	0.97	0.91	0.87	0.81	0.75	0.82	0.8	0.77	0.73	0.67	0.85	0.69	0.62	0.59	0.56
50	0.78	0.77	0.78	0.79	0.79	0.99	0.96	0.93	0.88	0.82	0.83	0.78	0.78	0.72	0.65	0.94	0.89	0.82	0.76	0.69
1.75 100	0.8	0.8	0.8	0.8	0.78	0.98	0.93	0.87	0.81	0.75	0.83	0.8	0.78	0.74	0.66	0.93	0.81	0.73	0.67	0.63
300	0.87	0.82	0.83	0.81	0.77	0.97	0.91	0.87	0.81	0.75	0.82	0.8	0.77	0.72	0.66	0.83	0.68	0.61	0.6	0.59
50	0.79	0.77	0.76	0.77	0.76	0.99	0.95	0.89	0.83	0.77	0.83	0.79	0.78	0.72	0.65	0.94	0.89	0.8	0.72	0.64
2 100	0.8	0.84	0.8	0.78	0.76	0.98	0.89	0.81	0.71	0.69	0.82	0.79	0.77	0.72	0.64	0.93	0.78	0.69	0.65	0.6
300	0.85	0.83	0.8	0.78	0.75	0.95	0.88	0.79	0.72	0.61	0.81	0.79	0.77	0.71	0.65	0.81	0.64	0.59	0.58	0.57
50	0.79	0.77	0.76	0.77	0.76	0.99	0.95	0.89	0.83	0.77	0.83	0.79	0.78	0.72	0.65	0.94	0.88	0.8	0.72	0.63
2.25 100	0.8	0.84	0.8	0.78	0.76	0.98	0.89	0.81	0.71	0.69	0.83	0.8	0.77	0.7	0.62	0.93	0.78	0.7	0.65	0.6
300	0.85	0.83	0.8	0.78	0.75	0.95	0.88	0.79	0.72	0.61	0.82	0.8	0.77	0.71	0.64	0.8	0.65	0.6	0.6	0.56
50	0.79	0.77	0.76	0.77	0.76	0.99	0.95	0.89	0.83	0.77	0.82	0.78	0.78	0.71	0.64	0.94	0.87	0.77	0.69	0.61
2.5 100	0.8	0.84	0.8	0.78	0.76	0.98	0.89	0.81	0.71	0.69	0.81	0.8	0.76	0.69	0.6	0.92	0.77	0.69	0.63	0.58
300	0.85	0.83	0.8	0.78	0.75	0.95	0.88	0.79	0.72	0.61	0.82	0.8	0.76	0.69	0.62	0.79	0.63	0.59	0.59	0.56
50	0.79	0.77	0.76	0.77	0.76	0.99	0.95	0.89	0.83	0.77	0.8	0.77	0.77	0.69	0.61	0.94	0.87	0.76	0.66	0.59
2.75 100	0.8	0.84	0.8	0.78	0.76	0.98	0.89	0.81	0.71	0.69	0.81	0.79	0.75	0.68	0.6	0.92	0.77	0.69	0.63	0.58
300	0.85	0.83	0.8	0.78	0.75	0.95	0.88	0.79	0.72	0.61	0.81	0.8	0.75	0.69	0.61	0.78	0.63	0.6	0.6	0.56
50	0.78	0.79	0.79	0.76	0.72	0.98	0.94	0.86	0.77	0.7	0.8	0.78	0.76	0.69	0.59	0.94	0.87	0.76	0.66	0.58
3 100	0.79	0.81	0.8	0.77	0.73	0.98	0.87	0.73	0.68	0.63	0.8	0.78	0.74	0.66	0.57	0.92	0.77	0.69	0.63	0.56
300	0.87	0.8	0.78	0.75	0.71	0.94	0.84	0.72	0.61	0.58	0.81	0.79	0.75	0.68	0.6	0.77	0.62	0.61	0.6	0.55
50	0.78	0.79	0.79	0.76	0.72	0.98	0.94	0.86	0.77	0.7	0.8	0.79	0.76	0.68	0.57	0.94	0.88	0.75	0.65	0.55
3.25 100	0.79	0.81	0.8	0.77	0.73	0.98	0.87	0.73	0.68	0.63	0.81	0.79	0.74	0.65	0.55	0.92	0.77	0.69	0.62	0.53
300	0.87	0.8	0.78	0.75	0.71	0.94	0.84	0.72	0.61	0.58	0.81	0.79	0.74	0.67	0.58	0.77	0.61	0.61	0.59	0.52
50	0.78	0.79	0.79	0.76	0.72	0.98	0.94	0.86	0.77	0.7	0.81	0.78	0.75	0.67	0.55	0.94	0.87	0.75	0.64	0.54
3.5 100	0.79	0.81	0.8	0.77	0.73	0.98	0.87	0.73	0.68	0.63	0.81	0.79	0.73	0.64	0.54	0.92	0.76	0.69	0.62	0.52
300	0.87	0.8	0.78	0.75	0.71	0.94	0.84	0.72	0.61	0.58	0.81	0.78	0.73	0.65	0.56	0.76	0.61	0.61	0.59	0.52

3.4.3.4 Homogeneous in-arborescence networks

Homogeneous in-arborescence networks under coordinated shocks exhibited a *sharp* increase in stability as the ratio $\frac{\mathcal{E}}{\mathcal{J}}$ of the total external asset to the total interbank exposure the system is increased beyond a particular threshold *provided the equity to asset ratio γ was approximately $\Phi/2$* . For example, for a 50-node homogeneous in-arborescence network under coordinated shock, ξ exhibited a sharp decrease from 0.76 to 0.18 for $\frac{\mathcal{E}}{\mathcal{J}} \in [0.75, 1]$, $\mathcal{K} = 0.1$, $\Phi = 0.5$ and $\gamma = 0.25 = \Phi/2$; see Figure 17 for a visual illustration.

To investigate the global extent of such a sharp decrease of ξ around a threshold value of $\frac{\mathcal{E}}{\mathcal{J}}$ in the range $[0.5, 1]$ with $\gamma \approx \Phi/2$, for each type of shocking mechanism, and for each values of the parameters n , Φ , $\gamma \approx \Phi/2$, and κ of the homogeneous in-arborescence network is computed, the ratio

$$\Delta(n, \Phi, \gamma, \mathcal{K}) = \frac{\max_{0.5 \leq \frac{\mathcal{E}}{\mathcal{J}} \leq 1} \{\xi\} - \min_{0.5 \leq \frac{\mathcal{E}}{\mathcal{J}} \leq 1} \{\xi\}}{\max_{\text{entire range of } \mathcal{E}} \{\xi\} - \min_{\text{entire range of } \mathcal{E}} \{\xi\}}$$

that provides the *maximum* percentage of the total change of the vulnerability index that occurred within this range of $\frac{\mathcal{E}}{\mathcal{J}}$. If the growth of ξ with respect to \mathcal{E}/\mathcal{J} was uniform or near uniform over the entire range of $\frac{\mathcal{E}}{\mathcal{J}}$, Δ would be approximately $\delta = \frac{1-0.5}{3.5-0.25} \approx 0.16$; thus, any value of Δ significantly higher than δ indicates a sharp transition within the above-mentioned range of \mathcal{E}/\mathcal{J} . As Table XXII shows, when $\gamma = \Phi/2$ a *significant* majority of the entries are coordinated shocks and many entries under idiosyncratic shocks are at least 2δ .

Formal intuition

A formal intuition behind such a sharp decrease of ξ can be provided as follows.

Lemma 3.4.6. *Fix γ , Φ , \mathcal{J} , a homogeneous in-arborescence network G and assume that $\gamma \approx \Phi/2$. Consider any node $v \in V_{\text{shock}}$ with $\text{deg}_{\text{in}}(v) > 1$, suppose that v fails due to the initial shock. Let u be any node such that u is a “leaf node” (i.e., $\text{deg}_{\text{in}}(u) = 0$), and $(u, v) \in E$. Then, as the total external asset \mathcal{E} of the network is varied, there exists a threshold value $\mathcal{E}_\tau(u)$ such that*

TABLE XXII. Values of $\Delta(n, \Phi, \gamma, \mathcal{K}) = \frac{\max_{0.5 \leq E/I \leq 1} \{\xi\} - \min_{0.5 \leq E/I \leq 1} \{\xi\}}{\max_{\text{entire range of } E} \{\xi\} - \min_{\text{entire range of } E} \{\xi\}}$ for homogeneous in-arborescence networks.

Entries that are at least 2×0.16 are shown in **boldface black**. Entries that are at least $\frac{3}{2} \times 0.16$ are shown in **boldface blue**

			\mathcal{K}										
			0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9		
Coordinated shock	V = 50	$\Phi = 0.5$	$\gamma = 0.25$	0.87	0.86	0.84	0.83	0.83	0.83	0.83	0.83	0.83	
			$\gamma = 0.3$	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$\Phi = 0.6$	$\gamma = 0.3$	0.27	0.49	0.52	0.52	0.52	0.52	0.52	0.52	0.52	0.52
			$\gamma = 0.35$	0.27	0.51	0.37	0.37	0.37	0.37	0.37	0.37	0.37	0.37
		$\Phi = 0.7$	$\gamma = 0.35$	0.27	0.48	0.59	0.64	0.65	0.65	0.65	0.65	0.65	0.65
	$\gamma = 0.4$		0.30	0.34	0.32	0.29	0.29	0.29	0.29	0.29	0.29	0.29	
	$\Phi = 0.8$	$\gamma = 0.4$	0.24	0.42	0.47	0.47	0.47	0.47	0.47	0.47	0.47	0.47	
		$\gamma = 0.45$	0.33	0.45	0.43	0.41	0.40	0.40	0.40	0.40	0.40	0.40	
	$\Phi = 0.9$	$\gamma = 0.45$	0.24	0.47	0.54	0.56	0.56	0.56	0.56	0.56	0.56	0.56	
		$\gamma = 0.5$	0.28	0.38	0.37	0.34	0.32	0.32	0.32	0.32	0.32	0.32	
	V = 100	$\Phi = 0.5$	$\gamma = 0.25$	0.95	0.94	0.94	0.93	0.93	0.93	0.93	0.93	0.93	
			$\gamma = 0.3$	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
		$\Phi = 0.6$	$\gamma = 0.3$	0.39	0.27	0.33	0.31	0.29	0.29	0.29	0.29	0.29	
			$\gamma = 0.35$	0.32	0.24	0.28	0.30	0.32	0.32	0.32	0.32	0.32	
		$\Phi = 0.7$	$\gamma = 0.35$	0.39	0.57	0.67	0.69	0.69	0.69	0.69	0.69	0.69	
	$\gamma = 0.4$		0.36	0.44	0.42	0.42	0.42	0.42	0.42	0.42	0.42		
	$\Phi = 0.8$	$\gamma = 0.4$	0.36	0.49	0.54	0.53	0.53	0.53	0.53	0.53	0.53		
		$\gamma = 0.45$	0.37	0.46	0.47	0.48	0.48	0.48	0.48	0.48	0.48		
$\Phi = 0.9$	$\gamma = 0.45$	0.39	0.56	0.66	0.67	0.67	0.67	0.67	0.67	0.67			
	$\gamma = 0.5$	0.37	0.47	0.48	0.47	0.47	0.47	0.47	0.47	0.47			
V = 300	$\Phi = 0.5$	$\gamma = 0.25$	0.93	0.94	0.94	0.93	0.93	0.93	0.93	0.93	0.93		
		$\gamma = 0.3$	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00		
	$\Phi = 0.6$	$\gamma = 0.3$	0.39	0.27	0.29	0.25	0.25	0.25	0.25	0.25	0.25		
		$\gamma = 0.35$	0.31	0.14	0.16	0.14	0.14	0.14	0.14	0.14	0.14		
	$\Phi = 0.7$	$\gamma = 0.35$	0.40	0.53	0.61	0.62	0.62	0.62	0.62	0.62	0.62		
$\gamma = 0.4$		0.34	0.36	0.34	0.33	0.33	0.33	0.33	0.33	0.33			
$\Phi = 0.8$	$\gamma = 0.4$	0.38	0.47	0.50	0.50	0.50	0.50	0.50	0.50	0.50			
	$\gamma = 0.45$	0.39	0.43	0.42	0.40	0.40	0.40	0.40	0.40	0.40			
$\Phi = 0.9$	$\gamma = 0.45$	0.40	0.53	0.60	0.61	0.61	0.61	0.61	0.61	0.61			
	$\gamma = 0.5$	0.39	0.44	0.44	0.43	0.43	0.43	0.43	0.43	0.43			
Idiosyncratic shock	V = 50	$\Phi = 0.5$	$\gamma = 0.25$	0.95	0.95	0.95	0.76	0.77	0.37	0.08	0.11	0.25	
			$\gamma = 0.3$	0.95	0.95	0.93	0.99	0.76	0.33	0.07	0.23	0.29	
		$\Phi = 0.6$	$\gamma = 0.3$	0.08	0.37	0.32	0.47	0.48	0.21	0.35	0.39	0.25	
			$\gamma = 0.35$	0.08	0.12	0.27	0.30	0.38	0.27	0.24	0.13	0.16	
		$\Phi = 0.7$	$\gamma = 0.35$	0.08	0.15	0.12	0.04	0.26	0.13	0.07	0.11	0.15	
	$\gamma = 0.4$		0.04	0.06	0.03	0.25	0.17	0.06	0.09	0.06	0.08		
	$\Phi = 0.8$	$\gamma = 0.4$	0.05	0.09	0.13	0.21	0.12	0.10	0.08	0.14	0.10		
		$\gamma = 0.45$	0.02	0.17	0.08	0.27	0.17	0.19	0.05	0.04	0.08		
	$\Phi = 0.9$	$\gamma = 0.45$	0.09	0.07	0.20	0.21	0.03	0.12	0.02	0.17	0.17		
		$\gamma = 0.5$	0.08	0.04	0.13	0.24	0.12	0.10	0.10	0.06	0.10		
	V = 100	$\Phi = 0.5$	$\gamma = 0.25$	0.98	0.96	0.98	0.96	0.67	0.37	0.16	0.03	0.16	
			$\gamma = 0.3$	0.99	1.00	0.95	0.99	0.71	0.31	0.07	0.11	0.26	
		$\Phi = 0.6$	$\gamma = 0.3$	0.04	0.15	0.3	0.44	0.03	0.08	0.22	0.16	0.06	
			$\gamma = 0.35$	0.03	0.04	0.29	0.30	0.17	0.33	0.06	0.10	0.10	
		$\Phi = 0.7$	$\gamma = 0.35$	0.02	0.11	0.09	0.23	0.13	0.01	0.10	0.05	0.06	
	$\gamma = 0.4$		0.02	0.1	0.16	0.11	0.12	0.13	0.05	0.03	0.05		
	$\Phi = 0.8$	$\gamma = 0.4$	0.03	0.03	0.07	0.07	0.07	0.04	0.06	0.02	0.08		
		$\gamma = 0.45$	0.03	0.06	0.14	0.17	0.09	0.13	0.08	0.03	0.08		
$\Phi = 0.9$	$\gamma = 0.45$	0.04	0.07	0.07	0.10	0.14	0.10	0.09	0.08	0.07			
	$\gamma = 0.5$	0.05	0.15	0.10	0.07	0.08	0.13	0.02	0.06	0.04			
V = 300	$\Phi = 0.5$	$\gamma = 0.25$	1.00	0.99	0.98	1.00	0.69	0.36	0.16	0.07	0.16		
		$\gamma = 0.3$	1.00	0.98	0.96	1.00	0.69	0.36	0.12	0.06	0.23		
	$\Phi = 0.6$	$\gamma = 0.3$	0.03	0.12	0.03	0.21	0.12	0.14	0.12	0.08	0.14		
		$\gamma = 0.35$	0.04	0.03	0.01	0.15	0.08	0.04	0.07	0.09	0.04		
	$\Phi = 0.7$	$\gamma = 0.35$	0.06	0.08	0.13	0.17	0.14	0.05	0.07	0.02	0.08		
$\gamma = 0.4$		0.03	0.06	0.09	0.11	0.10	0.08	0.02	0.02	0.05			
$\Phi = 0.8$	$\gamma = 0.4$	0.04	0.09	0.12	0.14	0.12	0.11	0.04	0.05	0.08			
	$\gamma = 0.45$	0.04	0.17	0.11	0.15	0.17	0.09	0.04	0.04	0.05			
$\Phi = 0.9$	$\gamma = 0.45$	0.05	0.08	0.11	0.18	0.14	0.02	0.03	0.04	0.07			
	$\gamma = 0.5$	0.05	0.07	0.10	0.21	0.11	0.09	0.04	0.01	0.05			

- if $\mathcal{E} < \mathcal{E}_\tau(u)$ then u will become insolvent, but
- if $\mathcal{E} > \mathcal{E}_\tau(u)$ then u will not become insolvent at any time $t \geq 1$, and the shock will not propagate any further through u .

Proof of Lemma 3.4.6

The amount of shock μ transmitted from v to u is given by

$$\mu = \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right) + \Phi \frac{\deg_{\text{out}}(v)}{\deg_{\text{in}}(v)}, 1 \right\}$$

Since G is an in-arborescence, $\deg_{\text{out}}(v) \leq 1$. First, consider the case of $\deg_{\text{out}}(v) = 0$. In this case,

$\mu = \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right), 1 \right\}$ and thus we have

$$c_u(1) = c_u(0) - \mu = \gamma \frac{\mathcal{E}}{n} - \min \left\{ (\Phi - \gamma) \left(1 + \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right), 1 \right\}$$

Assuming $\gamma \approx \Phi/2$, we have

$$\begin{aligned} c_u(1) &\approx \gamma \frac{\mathcal{E}}{n} + \deg_{\text{in}}(u) - \min \left\{ \gamma \left(1 + \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right), 1 \right\} \\ &= \min \left\{ \gamma \left(\frac{\mathcal{E}}{n} - 1 - \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right), \gamma \frac{\mathcal{E}}{n} - 1 \right\} \end{aligned}$$

There are two cases to consider:

- If $\gamma \left(\frac{\mathcal{E}}{n} - 1 - \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right) \geq \gamma \frac{\mathcal{E}}{n} - 1$ then

$$c_u(1) \approx \gamma \left(\frac{\mathcal{E}}{n} - 1 - \frac{\mathcal{E}}{n \deg_{\text{in}}(v)} \right) = \gamma \left(\frac{\mathcal{E}}{n} \left(1 - \frac{1}{\deg_{\text{in}}(v)} \right) - 1 \right)$$

Thus, if $\mathcal{E} > \mathcal{E}_{\tau_1}(u) = \frac{n}{1 - \frac{1}{\deg_{\text{in}}(v)}}$ then $c_u(1)$ would be strictly positive, the node u will not become insolvent at time $t = 1$, but if $\mathcal{E} < \mathcal{E}_{\tau_1}(u)$ then $c_u(1)$ would be strictly negative and u would fail.

- Otherwise, $c_u(1) \approx \gamma \frac{\mathcal{E}}{n} - 1$. Thus, if $\mathcal{E} > \mathcal{E}_{\tau_2}(u) = \frac{n}{\mathcal{E}}$ then $c_u(1)$ would be strictly positive, the node u will not become insolvent at time $t = 1$, but if $\mathcal{E} < \mathcal{E}_{\tau_1}(u)$ then $c_u(1)$ would be strictly negative and u would fail.

A similar analysis may be carried out if $\deg_{\text{out}}(v) = 1$ leading to *slightly* two different threshold values, say $\mathcal{E}_{\tau_3}(u)$ and $\mathcal{E}_{\tau_4}(u)$. Since $\deg_{\text{out}}(u) = 1$, if u does not become insolvent at time $t = 1$ then it does not become insolvent for any $t > 1$ as well.

The next lemma provides a lower bound, using the degree distributions of the Barábasi-Albert preferential-attachment model (12), on the expected value of the number of leaves in a random in-arborescence network for which Lemma 3.4.6 can be applied.

Lemma 3.4.7. *Consider a random in-arborescence $G = (V, E)$ generated by the Barábasi-Albert preferential-attachment algorithm (12) as outlined in Section 3.3.1 and let*

$$\widehat{V} = \left\{ u \in V \mid (\deg_{\text{in}}(u) = 0) \wedge (\exists v: ((\deg_{\text{in}}(v) > 1) \wedge ((u, v) \in E))) \right\}$$

Then, $\mathbb{E} \left[|\widehat{V}| \right] \geq \frac{n}{8} - \frac{11}{8}$.

Proof of Lemma 3.4.7

Let r be the root node of G . Note that, for any node $u \in V \setminus \{r\}$, $\deg_{\text{out}}(u) = 1$. Thus, using the results in (12), it follows that for any node $u \in V \setminus \{r\}$, $\Pr[\deg_{\text{in}}(u) = k - 1] \propto 1/k^3$ and in particular $\Pr[\deg_{\text{in}}(u) = 1] \leq 1/4$. For $j = 0, 2, \dots, n$, let n_j be the number of nodes u of G with $\deg_{\text{in}}(u) = j$.

Thus, $n = \sum_{j=0}^n n_j$, $|E| = n - 1 = \sum_{j=1}^n j n_j$, and

$$\begin{aligned} \sum_{u \in V \setminus \{r\}} \Pr[\deg_{\text{in}}(u) = 1] &\leq \mathbb{E}[n_1] \leq 1 + \sum_{u \in V \setminus \{r\}} \Pr[\deg_{\text{in}}(u) = 1] \\ &\equiv \frac{n-1}{4} \leq \mathbb{E}[n_1] \leq 1 + \frac{n-1}{4} \equiv \mathbb{E}[n_1] = \frac{n-1}{4} + t \quad \text{for some } t \in [0, 1] \end{aligned}$$

Letting $n_{>1} = \sum_{j=2}^n n_j$, we have

$$\mathbb{E}[n_0 + n_{>1}] = n - \mathbb{E}[n_1] = \frac{3n+1}{4} - t$$

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^n j n_j\right] &= n-1 \equiv \mathbb{E}[n_1] + \mathbb{E}\left[\sum_{j=2}^n j n_j\right] = n-1 \\ &\equiv \mathbb{E}[n_1] + 2\mathbb{E}[n_{>1}] \leq n-1 \Rightarrow \mathbb{E}[n_{>1}] \leq \frac{n-1 - \frac{3n+1}{4} + t}{2} = \frac{n}{8} + \frac{4t-5}{8} \end{aligned}$$

$$\mathbb{E}[n_0] = n - \mathbb{E}[n_1] - \mathbb{E}[n_{>1}] \geq n - \left(\frac{n-1}{4} + t\right) - \left(\frac{n}{8} + \frac{4t-5}{8}\right) = \frac{n}{8} + \frac{7}{8} - \frac{3t}{2}$$

and hence $\mathbb{E}[|\widehat{V}|]$ can be bounded as

$$\begin{aligned} \mathbb{E}[|\widehat{V}|] &= \mathbb{E}[n_0] - \mathbb{E}\left[\left|\left\{u \in V \mid (\deg_{\text{in}}(u) = 0) \wedge (\exists v: ((\deg_{\text{in}}(v) = 1) \wedge ((u, v) \in E)))\right\}\right|\right] \\ &\geq \mathbb{E}[n_0] - \mathbb{E}[n_1] \geq \left(\frac{n}{8} + \frac{7}{8} - \frac{3t}{2}\right) - \left(\frac{n-1}{4} + t\right) = \frac{n}{8} + \frac{9}{8} - \frac{5t}{2} \geq \frac{n}{8} - \frac{11}{8} \end{aligned}$$

Let $\xi(\mathcal{E})$ be the value of ξ parameterized by \mathcal{E} (keeping all other parameters unchanged), and let

$$\mathcal{E}_{\tau_{\min}} = \min \left\{ \mathcal{E}_{\tau}(u) \mid \deg_{\text{in}}(u) = 0, (u, v) \in E \text{ and } \deg_{\text{in}}(v) > 1 \right\}$$

$$\mathcal{E}_{\tau_{\max}} = \max \left\{ \mathcal{E}_{\tau}(u) \mid \deg_{\text{in}}(u) = 0, (u, v) \in E \text{ and } \deg_{\text{in}}(v) > 1 \right\}$$

It then follows that

$$\mathbb{E}[\xi(\mathcal{E}_{\tau_{\min}})] - \mathbb{E}[\xi(\mathcal{E}_{\tau_{\max}})] \geq \frac{\frac{n}{8} - \frac{11}{8}}{n} \approx \frac{1}{8}$$

and $\xi(\mathcal{E})$ exhibits a sharp decrease around the range $[\mathcal{E}_{\tau_{\min}}, \mathcal{E}_{\tau_{\max}}]$. In practice, the extent of this decrease is expected to be much more than the pessimistic lower bound of $1/8$, as the simulation results clearly show.

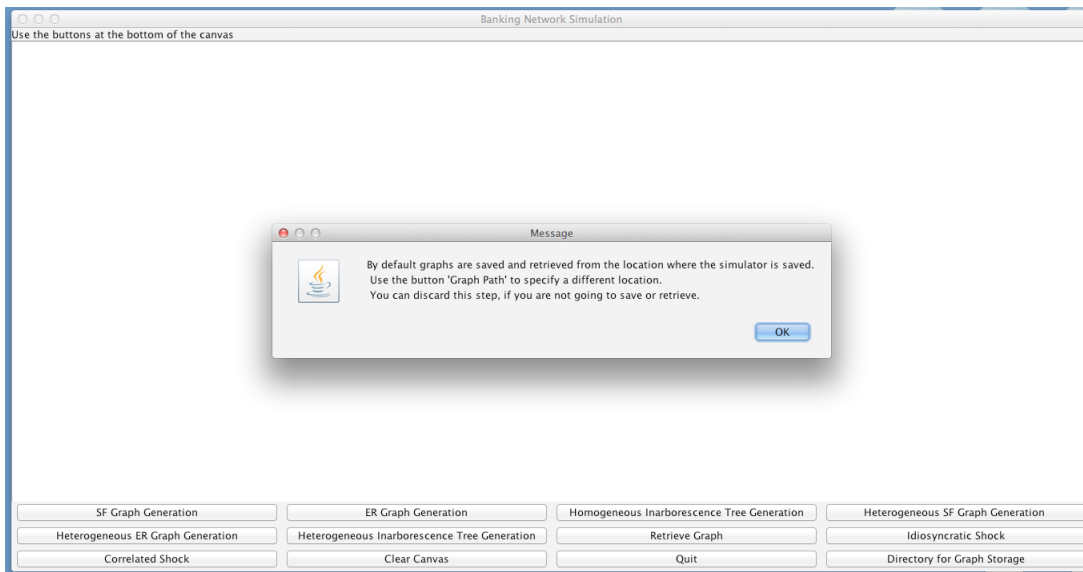


Figure 25. Initial Screen of the Simulator FIN-STAB.

3.5 Description of Simulator for Financial Stability(FIN-STAB)

When the user runs the simulator, a message box opens up (see Figure 25). The user may close the message box by selecting the **ok** button. Initially, the canvas is empty with a few buttons at the bottom. In order to study the effect of shocking a network, at least the following tasks need to be performed.

- The canvas has to be loaded with the network.
- Nodes in the network need to be shocked.

At any point of time:

- The user may clear the contents in the canvas by using the **Clear Canvas** button.
- The user may exit the simulator by selecting **Quit** button.

By default, the results of the simulator are stored in the current directory, i.e., the folder where the simulator is saved. The user can choose select different location to save the result by using the **Directory of Graph**

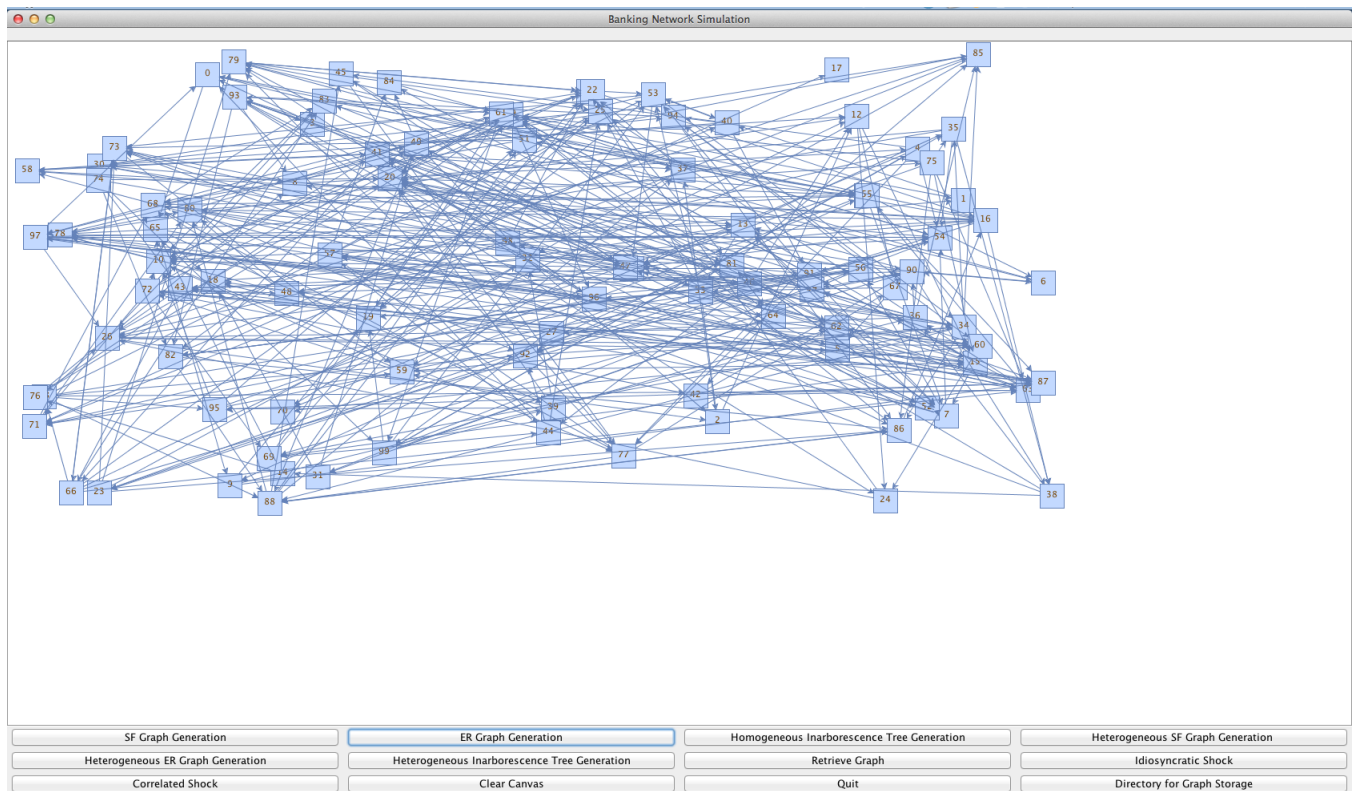


Figure 26. Homogeneous ER Graph with 100 nodes and Average degree 4.

Storage button, at which point the selected location becomes the new default location. The user will always be asked for the location to store the result once the result is ready.

3.5.1 Loading a network

A network can be loaded (see Figure 26) to the canvas either by

- Generating the network, or
- Retrieving an available network.

3.6 Description of Simulator for Financial Stability(FIN-STAB)

3.6.0.1 Generating a network

There are six different types of networks that can be generated by using the following buttons:

- Homogeneous Scale free network **SF Graph Generation**
- Homogeneous Erdos-Renyi random network **ER Graph Generation**
- Homogeneous inarborescence Tree **Homogeneous Inarborescence Tree Generation**
- Heterogeneous Scale free network **Heterogeneous SF Graph Generation**
- Heterogeneous Erdos-Renyi random network **Heterogeneous ER Graph Generation**
- Heterogeneous inarborescence tree **Heterogeneous Inarborescence Tree Generation**

The simulator requests the user for the following additional appropriate inputs and then generates the network.

Number of nodes of the network

This is the first step while generating any network. The default value is 100 (i.e., if the user does not enter the number of nodes then the simulator assumes the it to be 100). The user may decide to **cancel** the network generation by selecting the cancel button on the *input window*.

Average degree (for ER and SF networks only)

Next, the simulator requests the user about the average degree for ER and SF networks.

- The user may select **Yes** to generate a network with particular degree.

If the user selects this option, then the a new input window appears to request the value from the user.

The user may enter any value greater than 0 and less than or equal to 6. The default value is set as 3 (i.e., if the user does not enter a value then the simulator assumes it to be 3).

- To generate a network of random degree the user may select **No**.

Distribution of external and internal assets (for heterogeneous networks only)

For heterogeneous networks, the user is requested to input the distribution of internal and external assets in the network. There are two choices of the distribution:

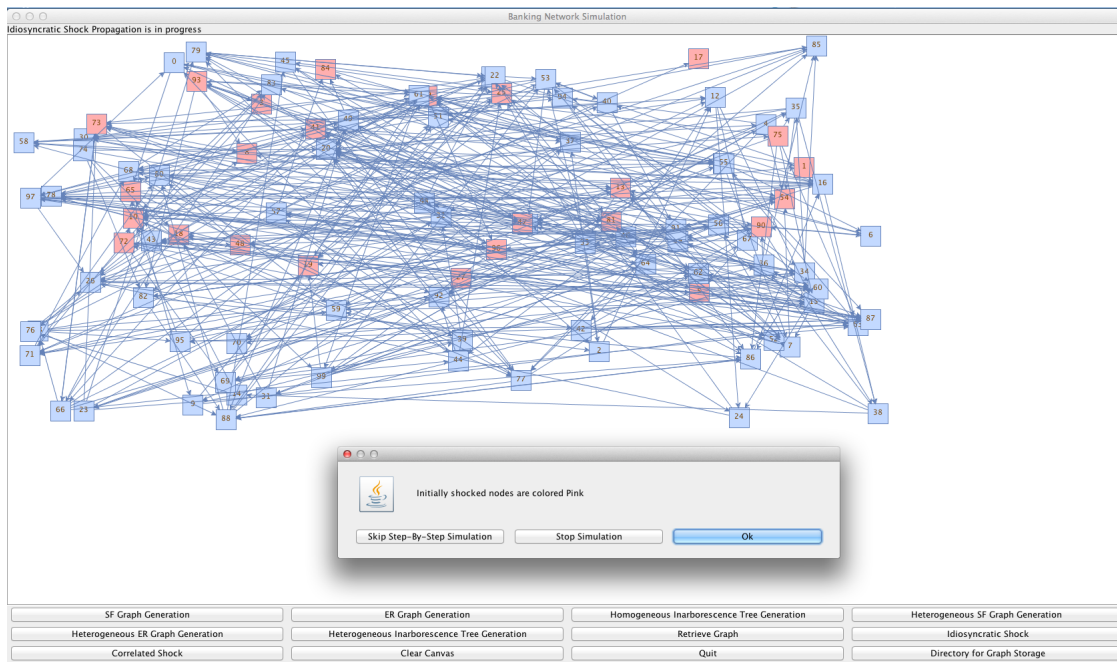


Figure 27. Homogeneous ER Graph with 100 nodes and Average degree 4.

- distribute 95
- distribute 60

Once the required details are entered the generated network appears on the canvas. A new window appears allowing the user to *save* the generated network. The user may proceed without saving the network. If the user decides to save the network a new window appears allowing the user to decide the location to save the network. The network is saved as the text file, and the user is asked for the name of this file.

3.6.0.2 Retrieving a network

The simulator allows the user to retrieve an available network (previously generated and saved, or manually generated by the user) from any location on the computer by selecting the **Retrieve Graph** button. The network is loaded on the canvas and is ready to be shocked.

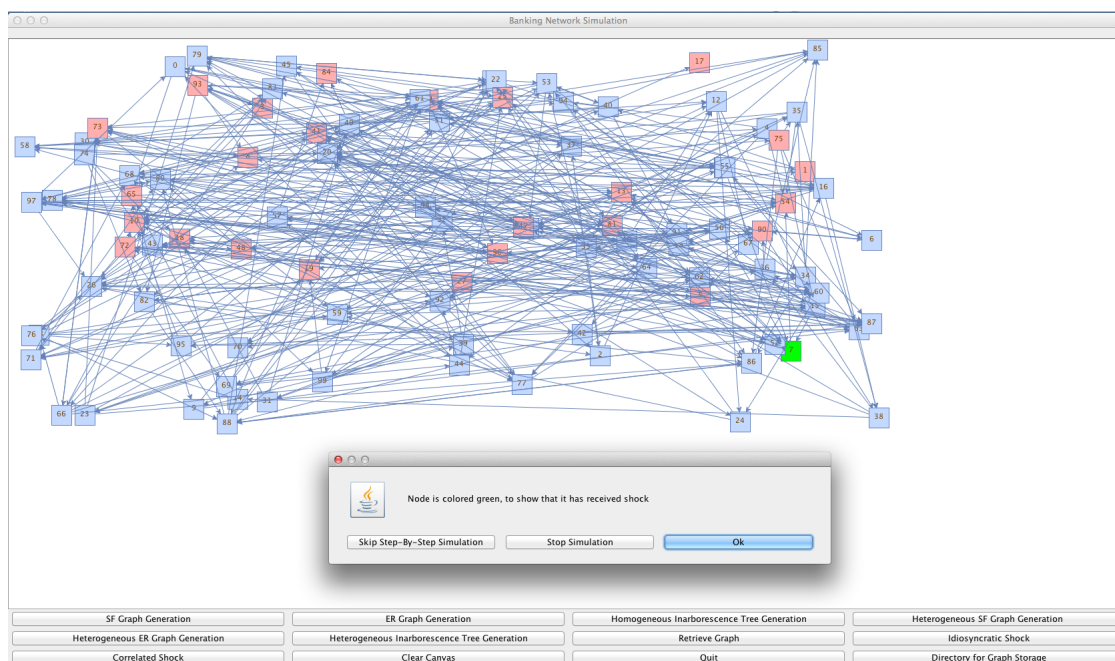


Figure 28. Shock propagation in Homogeneous ER Graph with 100 nodes and Average degree 4.

3.6.1 Shocking the network

There are two types of shock that can be applied to the graph

Idiosyncratic shock (shock randomly selected nodes)

Coordinated shock (a type of correlated shock)

Once the user selects the type of shock, the simulator requests for following inputs from the user:

Severity of the shock: a real number between 0 and 1, with a default value is 1/2.

The Equity to asset ratio: any positive real number that is less than the severity of the shock.

The External Assets to Internal Assets Ratio

The percentage of nodes to be shocked

Once the network is loaded and a shock is applied, the nodes initially shocked are colored in **pink**.

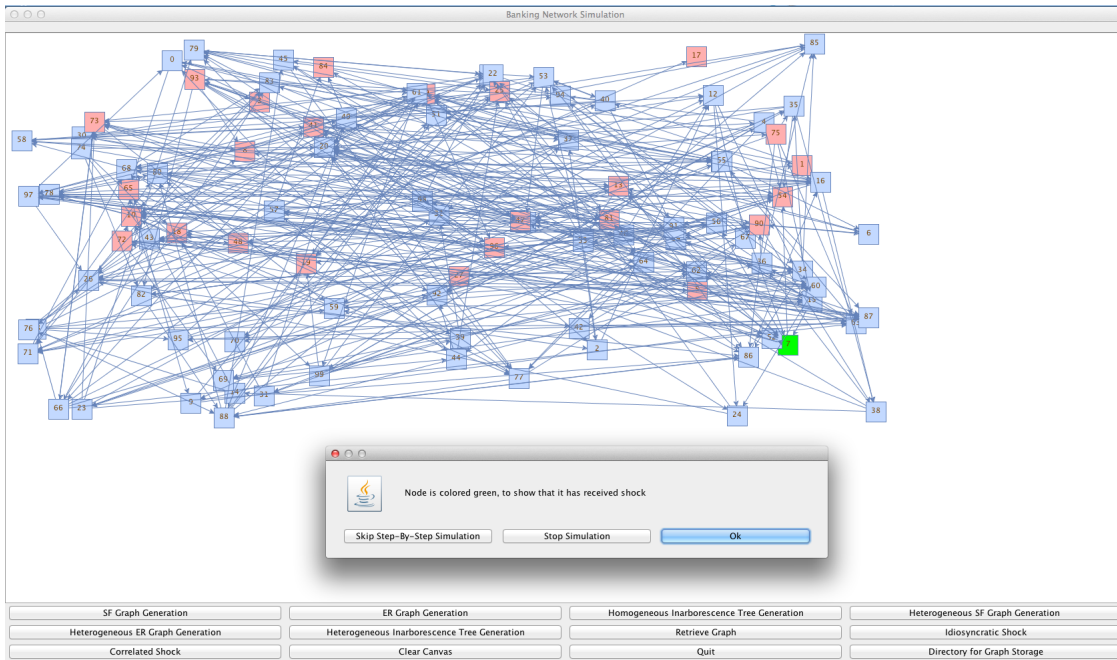


Figure 29. Shock propagation in Homogeneous ER Graph with 100 nodes and Average degree 4.

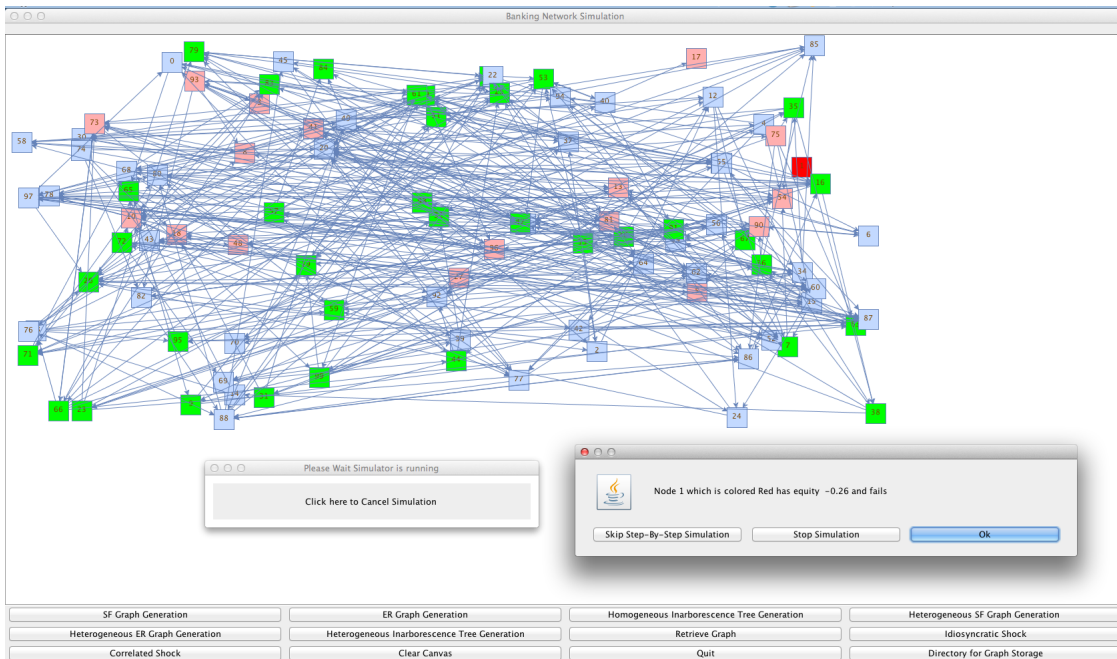


Figure 30. First node failed in Homogeneous ER Graph with 100 nodes and Average degree 4.

The user is now given three options:

- Continue step-by-step simulation (by selecting **OK**).
- Stop simulation.
- Stop step-by-step simulation skips the step-by-step propagation of shock and gives the final output.

When the nodes receiving the shock manage to survive the simulation is complete and the output window appears. If a node receiving the shock do not survive then it will propagate a part of the shock to its neighbors. The neighbors that receive the propagated shock are colored **green**. When a node receives a shock from its neighbor a window appears allowing the user to decide among the following three options:

- Continue step-by-step simulation (by selecting **OK**).
- Stop simulation.
- Stop step-by-step simulation skips the step-by-step propagation of shock and gives the final output.

If the user chooses to continue then the shock propagation continues. There is a window that appears during this shock propagation process informing the user that the simulator is active and also gives the user a opportunity to stop/cancel the simulation process. Each time a node fails a new window appears with the information about the failed node. The window also gives the user the following three options: decide among the following three options:

- Continue step-by-step simulation (by selecting **OK**).
- Stop simulation.
- Stop step-by-step simulation skips the step-by-step propagation of shock and gives the final output.

Once the shock propagation process is complete the output window appears. The user must close the output window. The user is now given a choice to either save the output or proceed without saving.

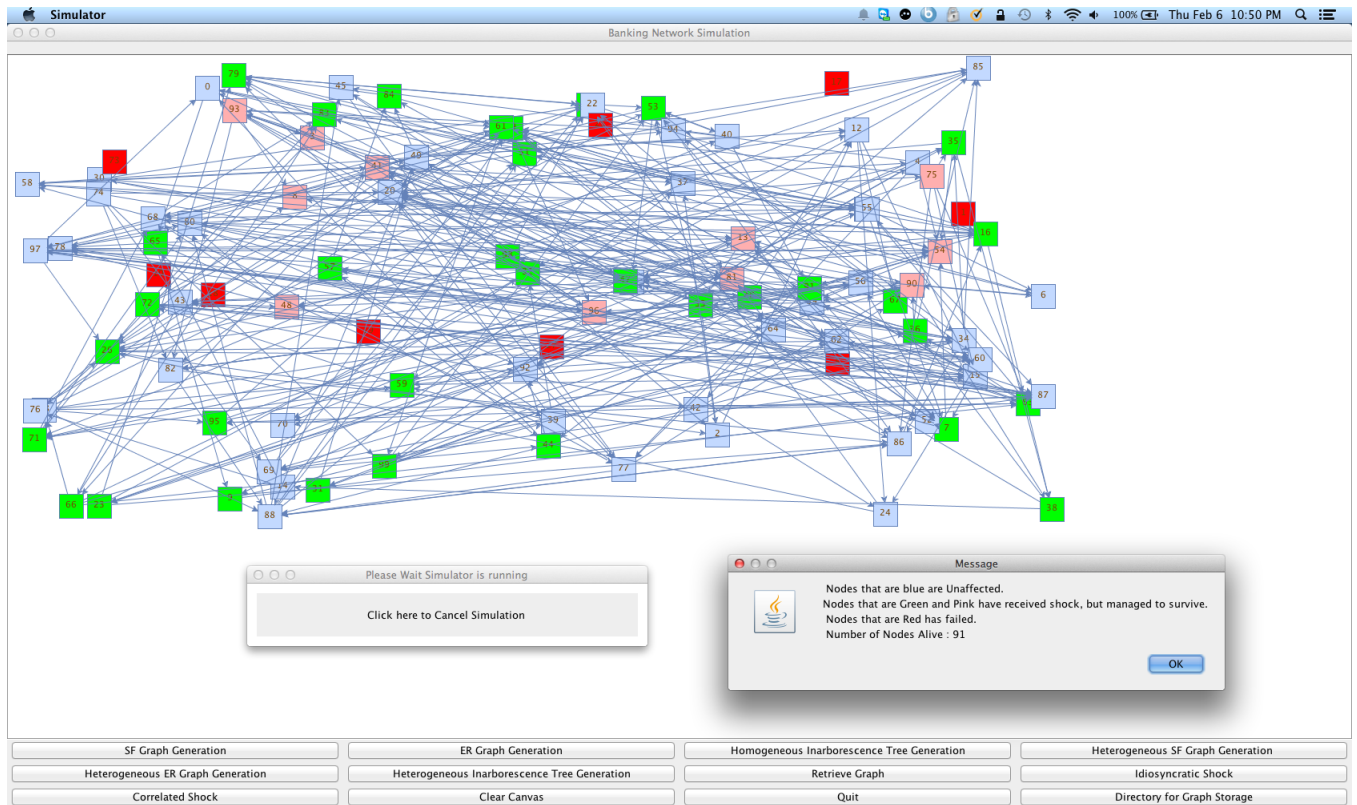


Figure 31. Homogeneous ER Graph with 100 nodes and Average degree 4. Shock propagation completed

3.7 Software Availability

An interactive software (called FIN-STAB) implementing an expanded version of the shock propagation algorithm is available from the website

<http://www2.cs.uic.edu/~dasgupta/financial-simulator-files>

CHAPTER 4

DENSELY ENTANGLED FINANCIAL SYSTEMS

In (91) A. Zawadoski introduces a banking network model in which the asset and counterparty risks are treated separately. The banks will hedge their assets risks by appropriate OTC contracts. In his model, each bank has only two counterparty neighbors (*i.e.*, the network is an undirected simple cycle), a bank fails due to the counterparty risk only if at least one of its two neighbors default, and such a counterparty risk is a low probability event. Informally, the author shows that the banks will hedge their asset risks by appropriate OTC contracts, and, though it may be socially optimal to insure against counterparty risk, in equilibrium banks will *not* choose to insure this low probability event.

In this chapter, we consider the above model for more general network topologies, namely when each node has exactly $2r$ counterparty neighbors for some integer $r > 0$. We extend the analysis of (91) to show that as the number of counterparty neighbors increase the probability of counterparty risk also increases, and in particular the socially optimal solution becomes privately sustainable when each bank hedges its risk to at least $n/2$ banks, where n is the number of banks in the network *i.e.*, when $2r$ is at least $n/2$, banks not only hedge their asset risk but also hedge its counterparty risk.

4.1 Related prior research works

As already mentioned, Zawadowski in (91) introduced a banking model in which asset risk and counterparty risk are treated separately, showed that banks always prefer to hedge their asset risk using OTC contracts and also showed that banks do not hedge their counter-party risk even though hedging counter-party risk is possible and socially desirable. Allen and Gale (6) showed that interbank deposits help banks share liquidity risk but expose them to asset losses if their counter-party defaults. Their model cannot be used to understand the contractual choices in case of OTC derivatives as they modeled the liquidity risk. Babus (11) proposed a model in which links are formed between banks which serves as an insurance mechanism to

reduce the risk of contagion. Allen and Babus (5) pointed out that graph-theoretic concepts provide a conceptual framework used to describe and analyze the banking network, and showed that more interbank links provide banks with a form of coinsurance against uncertain liquidity flows. Gai and Kapadi (50) showed that more interbank links increase the opportunity for spreading failures to other banks during crisis. Several prior researchers such as (5; 40; 64; 78) commented that graph-theoretic frameworks may provide a powerful tool for analyzing stability of banking and other financial networks. Kleindorfer *et al.* (64) argued that network analysis can play a crucial role in understanding many important phenomena in finance. Freixas *et al.* (47) explored the case of banks that face liquidity fluctuations due to the uncertainty about consumers withdrawing funds. Iazzetta and Manna (53) analysed the monthly data on deposit exchange to understand the spread of liquidity crisis using network topology.

Babus (10) studied how the trade-off between the benefits and the costs of being linked changes depending on the network structure and observed that, when the network is maximal, liquidity can be redistributed in the system to make the risk of contagion minimal. Corbo and Demange (33) explored the relationship of the structure of interbank connections to the contagion risk of defaults given the exogenous default of set of banks. Nier *et al.* (78) explored the dependency of systemic risks on the structure of the banking system via network theoretic approach and the resilience of such a system to contagious defaults. Haldane (53) suggested that contagion should be measured based on the interconnectedness of each institution within the financial system. Liedorp *et al.* (70) investigated if interconnectedness in the interbank market is a channel through which banks affect each others riskiness, and argued that both large lending and borrowing shares in interbank markets increase the riskiness of banks active in the dutch banking market. Kiyotaki and Moore (63) studied the chain reaction caused by the shock in one firm and the financial difficulties in other firms due to this chain reaction. Acharya and Bisin (3) compared centralized markets to OTC markets and showed that counter-party risk externalities can lead to excessive default and production of aggregate risk. Caballero and Simsek (23) concluded that OTC derivatives are not the sole reason for the inefficiency

of financial networks. Pirrong (80) argued that central counter-parties (CCP) can also increase the systemic risk under certain circumstances and hence the introduction of CCP will not guarantee to mitigate the systemic risk. Zawadowski (92) showed that complicated interwoven financial intermediation can be a reason for inefficient financial networks, and hence OTC are not the only reason for financial instability. Stulz (89) showed that exchange trading has both benefits and costs compared to OTC trading, and argued that credit default swaps (CDS) did not cause the crisis since they worked well during much of the first year of the crisis. Zhu and Pykhtin (93) showed that modeling credit exposures is vital for risk management application, while modelling credit value adjustment (CVA) is necessary step for pricing and hedging counter-party credit risk. Corbo and Demange (33) showed that introduction of central clearing house for credit default swaps will mitigate the counter-party risk. Gandhi *et al.* (52) paralleled and complemented the conclusion of (33), *i.e.*, the creation of central clearing house for CDS contracts may not reduce the counter-party risk.

4.2 The basic model

The model used here is different from the one used in previous chapter. The model has $n > 3$ banks and three time periods $t = 0, 1, 2$ termed as *initial*, *interim* and *final*, respectively. Each bank has exactly $2r$ counter-party neighbors for some integer $r > 0$ (see Fig. 1 for an illustration). The unit investment of each bank in the *long term* real asset yields a return of $R + \sum_{k=|i-1|}^{|i-r|} \varepsilon_k - \sum_{k=|i-(r+1)|}^{|i-2r|} \varepsilon_k$ at $t = 2$, where $R = \begin{cases} R_H, & \text{if the project succeeds} \\ R_L < R_H, & \text{if the project fails} \end{cases}$, and each ε_k is realized at $t = 2$ taking values of u or $-u$ each with probability $1/2$. For each unit investment made by the bank at $t = 0$, the investor lends $D \geq 0$ as short term debt and equity $1 - D \geq 0$ is the bank's share. The short term debt has to be rolled over at time period $t = 1$ for the banks to operate successfully. Thus the debt holders have an option to withdraw funding and force the bank to liquidate the real project.

Let $e \in \{0, 1\}$ be the unobservable effort choice such that a bank needs to exert an effort of $e = 1$ at both time period $t = 0$ and $t = 1$ for the project to be successful (*i.e.*, $R = R_H$). At $t = 1$ the project can be in one of the two states: a "bad" state with probability p or a "good" state with probability $1 - p$, irrespective

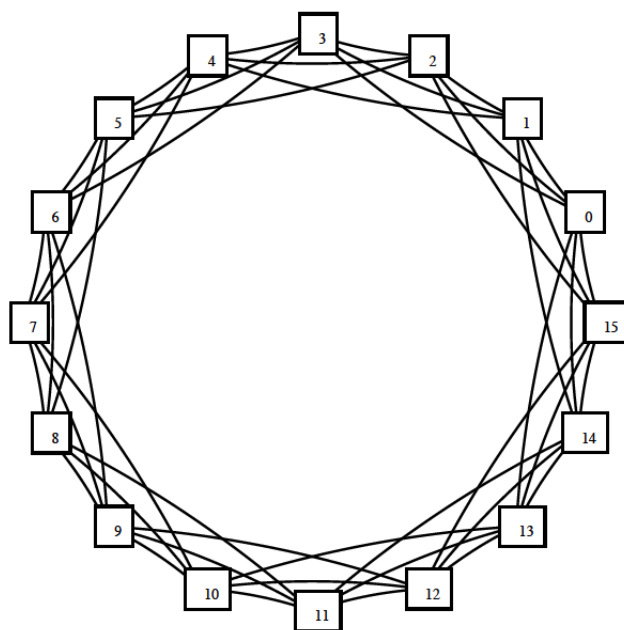


Figure 32. An illustration of a network of 16 banks $0, 1, \dots, 15$ with $r = 3$. Each i is connected to the banks $i-3(\text{mod } 16), i-2(\text{mod } 16), i-1(\text{mod } 16), i+1(\text{mod } 16), i+2(\text{mod } 16)$ and $i+3(\text{mod } 16)$.

of the effort exerted by the bank. At a “bad” state the project of one of the randomly chosen bank fails and delivers R_L , even if $e = 1$ at both time periods $t = 0$ and $t = 1$. Unless the bank demand collateral from its counterparties, if the bank defaults at $t = 1$ then all the hedging liabilities of the defaulted bank gets cancelled, the investors liquidate the bank and take equal share of L (the value of the bank when it is liquidated). If the bank survives till $t = 2$ and the counter-party risk gets realized then, the bank has to settle the counter-party hedging contract *before* paying its debt.

The following notations are used for four specific values of the probability of bad state p :

p^{soc} : if $p < p^{\text{soc}}$, then irrespective of the number of counter-party neighbors there is no need for counter-party insurance even from social perspective.

p^{ind} : if $p > p^{\text{ind}}$ then the banks will not buy counter-party insurance as the private benefits of insuring exceeds the cost.

p^{term} : if $p < p^{\text{term}}$ then the banks will continue to prefer short term debt.

p^{aut} : if $p < p^{\text{aut}}$ then no bank will have an incentive to hold more equity and borrow less.

Parameter Restrictions and Assumptions

The following parameter restriction are adopted from (91) to make them consistent for a network model with $2r$ counterparties. B is the banks' private benefit with the subscript representing the specific time period and X denotes the additional non-pledgable payoff. Inequality (Equation 4.1) ensures that the investors will choose to roll over the debt at $t = 1$ when the project is expected to succeed (*i.e.*, $R = R_H$), and the investors will decide to liquidate the bank at $t = 1$ if the bank's project is expected to fail (*i.e.*, $R = R_L$). Inequality (Equation 4.2) implies that it is socially optimal to exert effort. Inequality (Equation 4.3) ensures that banks have to keep positive equity to overcome moral hazard. Inequality (Equation 4.4) ensures that counter-party risk of the bank is large enough to lead to contagion but small enough that the bank does not want to engage in risk-shifting.

$$R_L < L < \left(1 - \frac{p}{n}\right)(R_H + X) - B_1 \quad (4.1)$$

$$R_H - R_L > \frac{2B_1}{1 - \frac{p}{n}} \quad (4.2)$$

$$B_1 \geq R_H - 1 + X \quad (4.3)$$

$$B_1 - X < u < \frac{B_1 \left(2^{2r} - \sum_{K=0}^r \frac{2r!}{K! (2r-K)!} \right)}{2r \sum_{K=0}^r \frac{2r-2K}{K! (2r-K)!}} \quad (4.4)$$

$$\beta > 1/2 \quad \text{and} \quad \beta > \frac{1 - (1-p)(R_H + X - B_1) - pL}{(1-p)B_1} \quad (4.5)$$

$$2u \leq B_0 < (1-p)B_1 \quad (4.6)$$

4.3 List of notations and variables

For the convenience of the reader we have included the notations and variables here.

n : Total number of banks in the network.

t : Three time periods are considered $t = 0, 1, 2$ which stands for initial, interim and final respectively.

β : Discount factor, assume $\beta < 1$.

R : Borrowing rate. R_0 is the borrowing rate at $t = 0$. R_1 is borrowing rate at $t=1$.

ϵ : It is the independent random variable realized at $t = 2$. It takes values $+u$ with probability $1/2$ and $-u$ with probability $1/2$.

L : If investments are liquidated early at $t = 1$ then the returns is only $L(< R_H)$.

X : Non-pledgeable payoff that is contingent.

r : In this model each bank is assumed to have $2r$ counterparty neighbors.

D : It is the investors share of investment in the bank at $t = 0$. $1 - D$ is the bankers share of investment in the bank at $t = 0$. $D \geq 0$ and $1 - D \geq 0$.

e : It is the unobservable effort choice made by the banker, where $e \in \{0, 1\}$.

B_i : Bankers private benefit at the time period i .

p : Probability of bad state.

p^{soc} : if $p < p^{\text{soc}}$, then irrespective of the number of counterparty neighbors there is no need for counterparty insurance even from social perspective.

p^{ind} : if $p > p^{\text{ind}}$ then the banks will not buy counterparty insurance as the private benefits of insuring exceeds the cost.

p^{term} : if $p < p^{\text{term}}$ then the banks will continue to prefer short term debt.

p^{aut} : if $p < p^{\text{aut}}$ then no bank will have an incentive to hold more equity and borrow less. $p^{\text{aut}} = \min\{p^{\text{s.aut}}, p^{\text{r.aut}}, p^{\text{f.aut}}\}$, where the superscripts s.aut , r.aut and f.aut stand for safe autarky, risky autarky and full autarky, respectively.

D^{max} : The maximum amount of borrowing at $t = 0$ that can be roll over at $t = 1$.

s^{safe} : The price per unit of default insurance in a stable system where all banks buy counterparty insurance.

D^{safe} : The amount borrowed by the bank for unit investment in the project in a stable system where all banks buy counterparty insurance.

D^* : The amount borrowed by the bank for unit investment in the project in a contagious system where banks decide not to buy insurance.

4.4 Glossary of financial terminology

Risk: Risk is a chance that an investment's actual return will be less than expected.

Assets: Assets are anything that is owned by an individual or business that has a monetary value.

Counterparty Risk: Risk that one party in an agreement defaults on its obligation to repay or return securities.

Hedging: A strategy that can be employed to reduce the risk by making a transaction in one market to protect against the loss in another.

Over The Counter(OTC): A market that is conducted between dealers by telephone and computer and not on a listed exchange. OTC stocks tend to be those of companies that do not meet the listing requirements of an exchange, although some companies that do meet the listing requirements choose to remain as OTC stocks. The deals and instruments are generally not standardized and there is no public record of the price associated with any transaction

Equity: It is the amount that shareholders own, in the form of common or preferred stock, in a publicly quoted company. Equity is the risk-bearing part of the company's capital and contrasts with debt

capital which is usually secured in some way and which has priority over shareholders if the company becomes insolvent and its assets are distributed

Liquidity: The degree to which an asset or security can be bought or sold in the market without affecting the asset's price. The ability to convert an asset to cash quickly.

Credit Default Swap(CDS): A specific kind of counterparty agreement that allows the transfer of third party credit risk from one party to the other. One party in the swap is a lender and faces credit risk from a third party, and the counterparty in the swap agrees to insure this risk in exchange for regular periodic payments.

4.5 Results

Results imply that when the number of counter-party neighbors is at least $n/2$, the socially optimal outcome become privately sustainable.

Theorem 4.5.1. *If the probability of bad state is $p \in [0, p^*)$, where $p^* = \min\{p^{\text{ind}}, p^{\text{aut}}, p^{\text{term}}\}$ then the followings hold.*

- (a) *Banks endogeneously enter into OTC contracts as shown by Zawadowski in (91).*
- (b) *Banks borrow $D < 1$ for short term at $t = 0$ at an interest rate $R_{t,0} > 1$ and $R_{t,1} = 1$ as shown by Zawadowski in (91).*
- (c) *In a bad state, failure of a single bank leads to run on all banks in the system only when $2r < n/2$.*
- (d) *If a bank loses at least r counterparties, it needs a debt reduction of $I = ru - B_1 + X > 0$.*
- (e) *The contagious equilibrium stated by Zawadowski in (91) exist only if $2r < n/2$. When $2r \geq n/2$ banks insure against counter-party risk using default insurance.*
- (f) *If $p \in (p^{\text{soc}}, p^{\text{ind}})$ then the socially optimal outcome is sustainable in equilibrium.*

Theorem 4.5.1(a) is proved in Lemma 4.5.9, whereas Theorem 4.5.1(c),(e),(f) are showed in Lemma 4.5.11. Theorem 4.5.1(d) follows from the derivations of parameter values as described in section 4.5.1 ; these derivations follow from the work in (91) and are provided in the appendix. Theorem 4.5.1(b) uses the same proof as that in (91) and is therefore omitted.

4.5.1 Proofs of theorem:

The derivations of the following parameters and their values described in items (I) - (VIII) below follows from the work in (91) since they are not affected by changing the number of counter-party neighbors from two to $2r$.

- (I) The maximum amount of borrowing at $t = 0$ that can be roll over at $t = 1$ is given by:

$$D^{\max}(R_{t,0}) = \frac{R_H - B_1 + X}{R_{t,0}} < 1$$

and the expected payoff of the above bank borrowing D^{\max} is B_1 .

- (II) In a stable system where all banks buy counter-party insurance, the price per unit of default insurance is

$$s^{\text{safe}} = \frac{1 - \beta}{n} - \beta \frac{p}{n}$$

(where the superscript “safe” denotes the insured system).

- (III) In a stable system where all banks buy counter-party insurance, the amount borrowed by the bank for unit investment in the project is

$$D^{\text{safe}} = \left(1 - \frac{p}{n}\right) (R_H - B_1 + X) + \frac{p}{n} L$$

(IV) In a contagious system where banks decide not to buy insurance, the amount borrowed by the bank for unit investment in the project is

$$D^* = (1 - p)(R_H - B_1 + X) + pL$$

(V) In a contagious system where banks decide not to buy insurance, the interest rate for the amount borrowed is

$$R_{i,0}^* = \frac{1}{1 - p \left(1 - \frac{L}{R_H - B_1 + X} \right)}$$

$$(VI) \quad p^{\text{term}} = (1 - \beta) \left(\frac{n}{n-1} \right) \left(\frac{B_0}{R_H + X - L - B_1(1 - \beta) + \frac{R_L + \beta B_1 - L}{n-1}} \right)$$

$$(VII) \quad p^{\text{f.aut}} = \frac{(1 - \beta)(R_H + X - B_1 - R_L)}{R_H + X - L - B_1(1 - \beta) - \frac{\beta}{n}(R_H - R_L)}$$

(VIII) $p^{\text{aut}} = \min \{ p^{\text{s.aut}}, p^{\text{r.aut}}, p^{\text{f.aut}} \}$, where the superscripts s.aut, r.aut and f.aut stand for safe autarky, risky autarky and full autarky, respectively.

Lemma 4.5.2. $p^{\text{ind}} = \frac{I(1 - \beta)}{\beta B_1}$

(Informally, Lemma 4.5.2 states that the probability of bad state must be at least $p^{\text{ind}} = \frac{I(1 - \beta)}{\beta B_1}$ for a bank's benefit to outweigh the cost of its counter-party insurance).

Proof. The private incentive of a single bank to stay insured instead of deviating from counter-party insurance is when the following holds:

$$\begin{aligned} & \text{benefits from all banks being insured} \geq \text{benefits from all but one bank not being insured} \\ & \equiv \beta \left(1 - \frac{(1 + 2r)p}{n} \right) B_1 + \beta \frac{2rp}{n} (B_1 + I) - s^{\text{safe}}(2rI) - (1 - D^{\text{safe}}) \geq \beta \left(1 - \frac{(1 + 2r).p}{n} \right) B_1 - (1 - D^{\text{safe}}) \end{aligned}$$

In the left -hand side of the inequality, the first term is the payoff if no bank defaults, the second term is the payoff if any of the counterparties of the bank default, the third term is money invested to buy the insurance on the bank's counterparties, the fourth term is the bank's equity.

$$\begin{aligned}
&\equiv \beta \frac{2rp}{n}(B_1 + I) - s^{\text{safe}}(2rI) \geq 0 \equiv \beta \frac{2rp}{n}(B_1 + I) - \left(\frac{1-\beta}{n} + \beta \frac{p}{n}\right)(2rI) \geq 0 \\
&\equiv \frac{2\beta rp B_1}{n} + \frac{2\beta rp I}{n} - \frac{2rI}{n} + \frac{2r\beta I}{n} - \frac{2\beta rp I}{n} \geq 0 \equiv \beta rp B_1 - rI + r\beta I \geq 0 \\
&\equiv \beta rp B_1 \geq rI - r\beta I \equiv \beta p B_1 \geq I - \beta I \equiv p \geq \frac{I(1-\beta)}{\beta B_1} \implies p^{\text{ind}} = \frac{I(1-\beta)}{\beta B_1}
\end{aligned}$$

□

Lemma 4.5.3. $p^{\text{soc}} = \frac{2Ir(1-\beta)}{(n-1)(R_H + X - L - B_1(1-\beta))}$.

(Informally, Lemma 4.5.3 states that the probability of bad state must be at least $p^{\text{soc}} = \frac{2Ir(1-\beta)}{(n-1)(R_H + X - L - B_1(1-\beta))}$ for a social benefit to outweigh the social cost of its counter-party insurance).

Proof. The social benefits outweigh the social cost of the counter-party insurance of the system if and only if the following holds:

benefits from all banks being insured \geq benefits from no bank being insured

$$\begin{aligned}
&\equiv \beta \left(1 - \frac{(1+2r)p}{n} B_1 + \beta \frac{2rp}{n} B_1 + I\right) - s^{\text{safe}}(2rI) - (1 - D^{\text{safe}}) \geq \beta(1-p)B_1 - (1 - D^*) \\
&\equiv \beta B_1 - \frac{\beta B_1 p}{n} - \frac{2\beta B_1 p r}{n} + \frac{2\beta B_1 p r}{n} + \frac{2\beta I p r}{n} - \left(\frac{1-\beta}{n} + \beta \frac{p}{n}\right)(2rI) - 1 + \left(1 - \frac{p}{n}\right)(R_H + X - B_1) + \frac{pL}{n} \\
&\quad \geq \beta B_1(1-p) - 1 + (1-p)(R_H + X - B_1) + pL \\
&\equiv \beta B_1 - \frac{\beta B_1 p}{n} + \frac{2\beta I p r}{n} - \frac{2rI}{n} + \frac{2\beta rI}{n} - \frac{2\beta I p r}{n} - 1 + R_H + X - B_1 - \frac{pR_H}{n} - \frac{pX}{n} + \frac{pB_1}{n} + \frac{pL}{n} \\
&\quad \geq \beta B_1 - p\beta B_1 - 1 + R_H + X - B_1 - pR_H - pX + pB_1 + pL
\end{aligned}$$

$$\begin{aligned}
&\equiv -\frac{\beta B_1 p}{n} - \frac{2rI}{n} + \frac{2\beta rI}{n} - \frac{pR_H}{n} - \frac{pX}{n} + \frac{pB_1}{n} + \frac{pL}{n} \geq -p\beta B_1 - pR_H - pX + pB_1 + pL \\
&\equiv -\frac{\beta B_1 p}{n} - \frac{pR_H}{n} - \frac{pX}{n} + \frac{pB_1}{n} + \frac{pL}{n} + p\beta B_1 + pR_H + pX - pB_1 - pL \geq \frac{2rI}{n} - \frac{2\beta rI}{n} \\
&\equiv p(-\beta B_1 - R_H - X + B_1 + L - \eta\beta B_1 + nR_H + nX - nB_1 - nL) \geq 2Ir(1-\beta) \\
&\equiv p(\beta B_1(n-1) + R_H(n-1) + X(n-1) - B_1(n-1) - L(n-1)) \geq 2Ir(1-\beta) \\
&\equiv p \geq \frac{2Ir(1-\beta)}{(n-1)(\beta B_1 + R_H + X - B_1 - L)} \equiv p \geq \frac{2Ir(1-\beta)}{(n-1)(R_H + X - L - B_1(1-\beta))}
\end{aligned}$$

$$\text{and thus } p^{\text{soc}} = \frac{2Ir(1-\beta)}{(n-1)(R_H + X - L - B_1(1-\beta))}. \quad \square$$

Corollary 4.5.4. *If each bank has only two counterparties then $r = 1$ and thus $p^{\text{soc}} = \frac{2I(1-\beta)}{(n-1)(R_H + X - L - B_1(1-\beta))}$.*

Lemma 4.5.5. (Probability of failure in risky autarky).

$$p^{\text{r.aut}} = (1-\beta) \frac{n}{n-1} X \frac{1 - \frac{2r!}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!}}{R_H + X - L - B_1(1-\beta) - \frac{(n\beta-1)X}{n-1} \left(1 - \frac{2r!}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!} \right)}.$$

(An informal explanation of 4.5.5 is as follows. If a bank has enough equity to survive even if all of its counter-parties collapse then the bank may be forced to borrow less and this is not profitable. But, if the bank decides to deviate to risky autarky then it cannot pay back its debt if the real project delivers $R_H - 2ru$.

The probability of failure if the bank chooses risky autarky is given by

$$p^{\text{r.aut}} = (1-\beta) \frac{n}{n-1} X \frac{1 - \frac{2r!}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!}}{R_H + X - L - B_1(1-\beta) - \frac{(n\beta-1)X}{n-1} \left(1 - \frac{2r!}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!} \right)}.$$

Proof. For a bank to be in risky autarky, they borrow less such that it can roll over the debt at $t = 1$ if its neighbors collapse but need not survive at $t = 2$. The relevant incentive constraint is:

$$R_H + \frac{2r! X}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!} - R_{l,0}^{\text{r.aut}} D^{\text{r.aut}} \geq B_1 \equiv R_{l,0}^{\text{r.aut}} D^{\text{r.aut}} \leq R_H - B_1 + \frac{2r! X}{2^{2r}} \sum_{K=0}^r \frac{1}{K!(2r-K)!}$$

The break-even condition for investors at $t = 0$ is:

$$D^{r,\text{aut}} = \frac{p}{n}L + \left(1 - \frac{p}{n}\right)R_{i,0}^{r,\text{aut}}D^{r,\text{aut}} \equiv D^{r,\text{aut}} = \frac{p}{n}L + R_H - B_1 - \frac{p}{n}R_H + \frac{p}{n}B_1 + \left(1 - \frac{p}{n}\right)\frac{2r!X}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}$$

A bank decides not to deviate to risky autarky in a contagious system if and only if the following holds:

$$\begin{aligned} & \text{payoff in contagious system} \geq \text{payoff in risky autarky} \\ & \equiv \beta(1-p)B_1 - (1-D^*) \geq \beta(1-p)\left(R_H + X - R_{i,0}^{r,\text{aut}}D^{r,\text{aut}}\right) + \beta\frac{n-1}{n}pB_1 - (1-D^{r,\text{aut}}) \\ & \equiv \beta B_1 - p\beta B_1 - 1 + R_H + X - B_1 - pR_H - pX + pB_1 \\ & \geq (\beta - p\beta)\left(R_H + X - R_H + B_1 - \frac{2r!X}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}\right) + \beta\frac{n-1}{n}pB_1 - 1 + \frac{p}{n}L \\ & \quad + R_H - B_1 - \frac{p}{n}R_H + \frac{p}{n}B_1 + \left(1 - \frac{p}{n}\right)\frac{2r!X}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!} \\ & \equiv \frac{n}{n-1}(1-\beta)X\left(1 - \frac{2r!}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}\right) \\ & \geq p\left[R_H + X - L - B_1(1-\beta) - \frac{(n\beta-1)X}{n-1}\left(1 - \frac{2r!}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}\right)\right] \end{aligned}$$

which implies

$$p^{r,\text{aut}} = (1-\beta)\frac{n}{n-1}X\frac{1 - \frac{2r!}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}}{R_H + X - L - B_1(1-\beta) - \frac{(n\beta-1)X}{n-1}\left(1 - \frac{2r!}{2^{2r}}\sum_{K=0}^r\frac{1}{K!(2r-K)!}\right)}. \quad \square$$

Corollary 4.5.6. *If the number of neighbors is 2 then $r = 1$ and thus*

$$\begin{aligned} p^{r,\text{aut}} &= (1-\beta)\frac{n}{n-1}X\frac{1 - \frac{2}{4}\left(\frac{1}{2} + \frac{1}{1}\right)}{R_H + X - L - B_1(1-\beta) - \frac{n\beta-1}{n-1}X\left(1 - \frac{2}{4}\left(\frac{1}{2} + \frac{1}{1}\right)\right)} \\ &\equiv p^{r,\text{aut}} = (1-\beta)\frac{n}{n-1}X\frac{1 - \frac{3}{4}}{R_H + X - L - B_1(1-\beta) - \frac{n\beta-1}{n-1}X\left(1 - \frac{3}{4}\right)} \\ &\equiv p^{r,\text{aut}} = (1-\beta)\frac{n}{n-1}\frac{\frac{X}{4}}{R_H + X - L - B_1(1-\beta) - \left(\frac{n\beta-1}{n-1}\right)\left(\frac{X}{4}\right)} \end{aligned}$$

Lemma 4.5.7. (Probability of failure in safe autarky).

$$p^{\text{s.aut}} = (1 - \beta) \left(\frac{n}{n-1} \right) \frac{2ru + X - B_1}{R_H + X - L - B_1 + \beta B_1 + \frac{1-\beta}{n-1}(X + 2ru - B_1)}.$$

(An informal explanation of Lemma 4.5.7 is as follows. If a bank chooses to deviate to safe autarky, it will survive unless it is directly affected by low returns R_L . It can pay back its debt even if the real project delivers $R_H - 2ru$. The probability of failure if the bank chooses safe autarky is given by $p^{\text{s.aut}} =$

$$(1 - \beta) \left(\frac{n}{n-1} \right) \frac{2ru + X - B_1}{R_H + X - L - B_1 + \beta B_1 + \frac{1-\beta}{n-1}(X + 2ru - B_1)}).$$

Proof. Suppose that a bank survives at $t = 2$ even if all of its counter-party risks are realized to ensure that the payoff of the real project is $R_H - 2ru$. Since the bank needs enough equity to survive even if real project yields $R_H - 2ru$, we have

$$R_H - 2ru - R_{i,0}^{\text{s.aut}} D^{\text{s.aut}} \geq 0 \equiv R_{i,0}^{\text{s.aut}} D^{\text{s.aut}} \leq R_H - 2ru$$

The break-even condition for investors at $t = 0$ is:

$$D^{\text{s.aut}} = \left(1 - \frac{p}{n}\right) R_{i,0}^{\text{s.aut}} D^{\text{s.aut}} + \frac{p}{n} L \equiv D^{\text{s.aut}} = \left(1 - \frac{p}{n}\right) (R_H - 2ru) + \frac{p}{n} L$$

Banks do not deviate to safe autarky from contagious system if and only if the following holds:

payoff in contagious system \geq payoff in safe autarky

$$\equiv \beta(1 - p)B_1 - (1 - D^*) \geq \beta \left(1 - \frac{p}{n}\right) (R_H + X - R_{i,0}^{\text{s.aut}} D^{\text{s.aut}}) - (1 - D^{\text{s.aut}})$$

$$\equiv \beta B_1 - p\beta B_1 - 1 + R_H + X - B_1 - pR_H - pX + pB_1 + pL$$

$$\geq \left(\beta - \frac{p\beta}{n}\right) (R_H + X - R_H + 2ru) - 1 + R_H - 2ru - \frac{p}{n}R_H + \frac{p}{n}(2ru) + \frac{p}{n}L$$

$$\equiv \beta B_1 - p\beta B_1 + X - B_1 - pR_H - pX + pB_1 + pL$$

$$\geq \beta R_H + \beta X - \beta R_H + 2\beta ru - \frac{p\beta R_H}{n} - \frac{p\beta X}{n} + \frac{p\beta R_H}{n} - \frac{2p\beta ru}{n} - 2ru - \frac{p}{n}R_H + \frac{p}{n}(2ru) + \frac{p}{n}L$$

$$\equiv \beta B_1 + X - B_1 - \beta X - 2\beta ru + 2ru \geq \frac{p}{n} \left(n\beta B_1 + nR_H + nX - nB_1 - nL - \beta X - 2r\beta u - R_H + 2ru + L \right)$$

$$\begin{aligned}
&\equiv (1-\beta)X - (1-\beta)B_1 + 2ru(1-\beta) \\
&\quad \geq \frac{p}{n} \left(n\beta B_1 + R_H(n-1) + nX - nB_1 - L(n-1) - \beta X + 2ru(1-\beta) + X - X + B_1 - B_1 + \beta B_1 - \beta B_1 \right) \\
&\equiv (1-\beta)n(2ru + X - B_1) \\
&\quad \geq p \left(\beta B_1(n-1) + R_H(n-1) + X(n-1) - B_1(n-1) - L(n-1) + X(1-\beta) + 2ru(1-\beta) - B_1(1-\beta) \right) \\
&\equiv (1-\beta)\frac{n}{n-1} (2Ru + X - B_1) \geq p \left(\beta B_1 + R_H + X - B_1 - L + \frac{1-\beta}{n-1}(X + 2ru - B_1) \right) \\
&\equiv p \leq (1-\beta) \left(\frac{n}{n-1} \right) \frac{2ru + X - B_1}{R_H + X - L - B_1 + \beta B_1 + \frac{1-\beta}{n-1}(X + 2ru - B_1)}
\end{aligned}$$

This implies $p^{\text{s.aut}} = (1-\beta) \left(\frac{n}{n-1} \right) \frac{2ru + X - B_1}{R_H + X - L - B_1 + \beta B_1 + \frac{1-\beta}{n-1}(X + 2ru - B_1)}$. □

Corollary 4.5.8. *If the number of neighbors is 2 then $r = 1$ and thus*

$$p^{\text{s.aut}} = (1-\beta) \left(\frac{n}{n-1} \right) \frac{2u + X - B_1}{R_H + X - L - B_1 + \beta B_1 + \frac{1-\beta}{n-1}(X + 2u - B_1)}$$

Lemma 4.5.9. *In equilibrium banks hedge all of its counter-party risks i.e. banks endogenously enter into OTC contracts.*

Proof. The banks hedge all of its counter-party risks if and only if

payoff from hedging in contagious equilibrium > payoff from not hedging

$$\begin{aligned}
&\equiv \beta(1-p)B_1 > \beta(1-p) \frac{2r!}{2^{2r}} \left(\frac{B_1 + 2ru}{0!(2r-0)!} + \frac{B_1 + (2r-2)u}{1!(2r-1)!} + \frac{B_1 + (2r-4)u}{2!(2r-2)!} + \dots + \frac{B_1 + (2r-2r)u}{\frac{2r}{2}! \left(2r - \frac{2r}{2}\right)!} \right) \\
&\equiv B_1 > \frac{2r!}{2^{2r}} \left(\frac{B_1 + 2ru}{0!(2r-0)!} + \frac{B_1 + (2r-2)u}{1!(2r-1)!} + \frac{B_1 + (2r-4)u}{2!(2r-2)!} + \dots + \frac{B_1 + (2r-2r)u}{\frac{2r}{2}! \left(2r - \frac{2r}{2}\right)!} \right) \\
&\equiv B_1 > \frac{2ur \sum_{K=0}^r \frac{2r-2K}{K!(2r-K)!}}{2^{2r} - \sum_{K=0}^r \frac{2r!}{K!(2r-K)!}}
\end{aligned}$$

which satisfies inequality (Equation 4.4), and thus banks hedge all of its counter-party risks.

Let the returns from the successful project be $R_H + \sum_{k=|i-1|}^{|i-r|} \varepsilon_k - \sum_{k=|i-(r+1)|}^{|i-2r|} \varepsilon_k$. Assume that the bank goes bankrupt at $t = 2$ if the counter-party realization of its unhedged risk is $-u$. This is true if the bank cannot repay its debt at $t = 2$, *i.e.*,

$$R_H - u < R_{t,0}^* D^* \equiv R_H - u < R_H - B_1 + X \equiv u > B_1 - X$$

if a bank fails when it loses $-u$ on its unhedged counter-party exposure, it will fail when the loss is greater than $-u$ on its unhedged counter-party exposure. \square

Corollary 4.5.10. *If number of counter-party neighbors is 2 (i.e., $r = 1$), then*

$$B_1 > \frac{2u \left(\frac{2-0}{0!(2-0)!} + \frac{2-2}{1!(2-1)!} \right)}{2^2 - \left(\frac{2!}{0!(2-0)!} + \frac{2!}{1!(2-1)!} \right)} \quad \text{and} \quad B_1 > 2u \equiv u > B_1 - X$$

Lemma 4.5.11. *When the number of neighboring counterparties is less than $n/2$, a bank chooses to shirk if one or more of its counterparties default by leaving the bank unhedged resulting in its debt not being rolled over at $t = 1$.*

Proof. If a bank borrows $D^{\max}(R_{t,0})$ at $t = 0$ and if the bank has a low expected realization of R_L , then the debt financing is not rolled over at $t = 1$. Since $R_L < L$, the creditors will want to terminate the project. The bank goes bankrupt if its debt financing is not rolled over at $t = 1$.

If $2r$, the number of counter-party neighbors, is less than $n/2$, then probability of failure due to counter-party risk is less than $\frac{n/2}{n}$, *i.e.*, the probability of counter-party risk is less than $1/2$. Since it is assumed that banks will consider a counter party risk probability of at least $1/2$ to insure against counter-party risk, banks do not insure against counter-party risk using default insurance when $2r < n/2$. When $2r \geq n/2$, the probability of counter-party risk becomes at least $1/2$ and hence banks will hedge the counter-party risk. The counter-party insurance pays off happens with probability $\frac{2rp}{n}$ in case of private perspective and with

probability $\frac{(n-1)p}{n}$ in case of social perspective. Thus, as $2r$ increases to $n - 1$, the counter-party insurance payoff probability in private perspective becomes the same as that in social perspective.

When $2r \geq n/2$, the individual banks will hedge the counter-party risk by taking counter-party insurance. When $2r < n/2$, banks will not hedge the counter-party risk and hence failure of a counter-party will lead to the violation of its incentive constraint, thus the bank shirks and the project delivers R_L . Let D_1 be the amount of debt to be rolled over at $t = 1$. The investors will demand higher interest rate $R_{i,1}$ in order to break even. Let P_s be the probability of a bank that do not default, P_f be the probability of a bank that defaults, and n_d be the number of neighbors of any bank that default. Thus, $P_s = \frac{2r - n_d}{2r}$ and $P_f = \frac{n_d}{2r}$. By the break even condition of investors, we get

$$P_s R_{i,1} D_1 + P_f R_L = D_1 \equiv R_{i,1} D_1 = \frac{D_1 - P_f R_L}{P_s}$$

The incentive constraint is

$$\begin{aligned} \beta(R_H - R_{i,1} D_1 + X) &\geq \beta B_1 \\ \equiv \beta \left(R_H - \frac{D_1 - P_f R_L}{P_s} + X \right) &\geq \beta B_1 \\ \equiv P_s R_H - D_1 + P_f R_L + P_s X &\geq B_1 P_s \\ \equiv P_s R_H - D_1 + P_f R_L + P_s X &\geq B_1 P_s \\ \equiv P_s (R_H + X - B_1) + P_f R_L &\geq D_1 \end{aligned}$$

If the bank had originally borrowed D_0 , the amount it has to roll over at $t = 1$ is $R_{i,0} D_0 = R_H + X - B_1$, but the amount that is actually getting rolled over is only $P_s (R_H + X - B_1) + P_f R_L$.

□

Corollary 4.5.12. *If number of counter-party neighbors is 2 (i.e., $2r = 2$) and number of banks defaulted is 1 (i.e., $n_d = 1$), then $P_s = \frac{2r - n_d}{2r} = 1/2$, $P_f = \frac{n_d}{2r} = 1/2$, and thus*

$$P_s (R_H + X - B_1) + P_f R_L \geq D_1 \equiv \frac{1}{2} (R_H + X - B_1) + \frac{1}{2} R_L \geq D_1$$

CHAPTER 5

CONCLUSION

In the first part of the thesis we investigated the systemic instabilities of the banking networks, an important component of modern capitalist economies of many countries. We have formalized a model for the shock propagation in a banking network model, defined the appropriate stability measure and provided their computational properties. We have performed a comprehensive empirical evaluation over more than 700,000 combinations of network types and parameter combinations. Our results and proofs also sheds some light on the properties of topologies and parameters of network that may lead to higher or lower stabilities. Flagging a network as vulnerable does not necessarily imply that such is the case, but such a network may require further analysis based on other aspects of free market economics that cannot be modeled. We view our as a necessary first step towards understanding vulnerabilities of banking systems due to sudden loss of external assets and hope that it will generate sufficient interests in both the banking network community and the network algorithms community to further investigate and refine the network model, stability issues and parameter choices. In future the research can be extended to

- Investigate the network model
 - Find a network structure that closely resembles real banking networks
 - Find an optimal architecture for a robust financial systems
- Study the effect of diversified external investments
- Modify stability as the percentage of the external assets that remain in the system at the end of shock propagation and study the stability of networks.
- Identify the modifications that can be made on the network topology or parameters, so that a system that is flagged vulnerable can be made into a stable system.

In the second part of the thesis we consider issues related to the financial risk management, which is a critical component of maintaining economic stability. We are using a different banking network model which is an extension of the model used in (91) by A. Zawadoski. In our model each banks has exactly $2r$ counter-party neighbors for some integer $r > 0$. We have shown that as the number of counter-party neighbors increase and becomes at least $n/2$ where n is the number of banks in the network, the banks will decide to insure not only their asset risk but also their counter-party risk. The research can be extended to analyze the means to protect the financial system from collapse when they are sparsely connected.

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