On optimal approximability results for computing the strong metric dimension☆

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Abstract

The strong metric dimension of a graph was first introduced by Sebö and Tannier (Mathematics of Operations Research, 29(2), 383-393, 2004) as an alternative to the (weak) metric dimension of graphs previously introduced independently by Slater (Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, 549-559, 1975) and by Harary and Melter (Ars Combinatoria, 2, 191-195, 1976), and has since been investigated in several research papers. However, the exact worst-case computational complexity of computing the strong metric dimension has remained open beyond being NP-complete. In this communication, we show that the problem of computing the strong metric dimension of a graph of n nodes admits a polynomial-time 2-approximation, admits a $O^*(2^{0.287\,n})$-time exact computation algorithm, admits a $O(1.2738^k + n\,k)$-time exact computation algorithm if the strong metric dimension is at most k, does not admit a polynomial time $(2-\varepsilon)$-approximation algorithm assuming the unique games conjecture is true, does not admit a polynomial time $(10\sqrt{5}-21-\varepsilon)$-approximation algorithm assuming P$\neq$ NP, does not admit a $O^*(2^{o(n)})$-time exact computation algorithm assuming the exponential time hypothesis is true, and does not admit a $O^*(n^{o(k)})$-time exact computation algorithm if the strong metric dimension is at most k assuming the exponential time hypothesis is true.

Keywords: Strong metric dimension, minimum node cover, approximability, unique games conjecture, exponential time hypothesis, parameterized complexity
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1. Introduction

The concept of the metric dimension of graphs was originally introduced independently by Slater [21] and by Harary and Melter [10] in the 1970's. Their definition involved determining a minimum number of nodes such that distance vectors from each of these nodes to all other nodes (the "resolving vectors") can be used to "distinguish" every pair of nodes in the graph. Computing the metric dimension is known to be NP-complete [9]. Optimal approximability

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results for the metric dimension was provided by Hauptmann *et al.* in [11] by showing both a $(\ln n + \ln\log_2 n + 1)$-approximation based on an approximation algorithm for test set problems in [2] and also a $(1-\varepsilon)$-inapproximability for any constant $0 < \varepsilon < 1$.

Unfortunately, the metric dimension of a graph suffers from two difficulties, namely that the problem does not provably admit a better-than-logarithmic approximation and the resolving vectors cannot be used to uniquely identify the graph. The **strong** metric dimension of a graph was therefore introduced by Sebö and Tannier [20] as an alternative to the above-mentioned metric dimension of graphs. The resulting "strongly" resolving vectors can indeed be used to uniquely identify the given graph. Subsequently, the strong metric dimension has been investigated in several research papers such as [18, 19, 25]. Let $G = (V, E)$ be a given undirected graph of n nodes. To define the strong metric dimension, we will use the following notations and terminologies:

- $\mathrm{N}(u) = \left\{ v \,\middle|\, \{u, v\} \in E \right\}$ denotes the set of neighbors of a node u.
- $u \overset{s}{\leftrightsquigarrow} v$ denotes a shortest path from between nodes u and v of length (number of edges) $d_{u,v}$.
- $\mathrm{diam}(G) = \max_{u,v \in V} \{ d_{u,v} \}$ denotes the diameter of a graph G.
- A shortest path $u \overset{s}{\leftrightsquigarrow} v$ is called *maximal*[1] if and only if it is not *properly* included inside another shortest path, *i.e.*, if and only if the predicate

$$\Big(\forall\, x \in \mathrm{N}(u) \colon d(x, v) \le d(u, v) \Big) \bigwedge \Big(\forall\, y \in \mathrm{N}(v) \colon d(y, u) \le d(u, v) \Big)$$

  is true.
- A node x *strongly resolves* a pair of nodes u and v, denoted by $x \blacktriangleright \{u, v\}$, if and only if either v is on a shortest path between x and u, or u is on a shortest path between x and v.
- A set of nodes $V' \subseteq V$ is a *strongly resolving set* for G, denoted by $V' \blacktriangleright G$, if and only if every distinct pair of nodes of G is strongly resolved by some node in V'.

Then, the problem of computing the strong metric dimension of a graph can be defined as follows:

Problem name:	Strong Metric Dimension (Str-Met-Dim)
Instance:	an undirected graph $G = (V, E)$.
Valid Solution:	a set of nodes $V' \subseteq V$ such that $V' \blacktriangleright G$.
Objective:	*minimize* $\lvert V' \rvert$.
Related notation:	$\mathrm{sdim}(G) = \min\limits_{V' \subseteq V \,\wedge\, V' \blacktriangleright G} \left\{ \left\lvert V' \right\rvert \right\}$.

1.1. Standard Concepts From the Algorithms Research Community

For the benefit of readers not familiar with analysis of approximation algorithms, we state below some standard definitions; see standard textbooks such as [8, 9, 23] for further details.

[1] The end-points of such a path is called a mutually maximally distant pairs of nodes in [20].

An algorithm for a minimization problem is said to have an *approximation ratio* of ρ (or simply called a ρ-approximation) provided the algorithm runs in polynomial time in the size of the input and produces a solution with an objective value *no larger than* ρ times the value of the optimum. A computational problem P is said to be ρ-inapproximable under a complexity-theoretic assumption of $\mathbb{A}$ provided, assuming $\mathbb{A}$ to be true, there exists no ρ-approximation for P. The (standard) Boolean satisfiability problem when every clause has exactly k literals will be denoted by k-Sat. Finally, for two functions $f(n)$ and $g(n)$ of n, we say $f(n) = O^*(g(n))$ if $f(n) = O(g(n)\, n^c)$ for some positive constant c.

1.2. Brief Overview of Three Well-known Complexity Theoretic Assumptions

For the benefit of those readers not well familiar with well-known complexity-theoretic assumptions, we provide a very brief overview of the three complexity-theoretic assumptions used in this communication.

The P≠NP assumption Starting with the famous Cook's theorem [4] in 1971 and Karp's subsequent paper in 1972 [14], the P≠NP assumption is the central assumption in structural complexity theory and algorithmic complexity analysis.

The Unique Games Conjecture (Ugc) The Unique Games Conjecture, formulated by Khot in [15], is one of the most important open question in computational complexity theory. Informally speaking, the conjecture states that, assuming P≠ NP, a type of constraint satisfaction problems does not admit a polynomial time algorithm to distinguish between instances that are almost satisfiable from instances that are almost completely unsatisfiable. There is a large body of research works showing that the conjecture has many interesting implications and many researchers routinely assume Ugc to prove non-trivial inapproximability results. An excellent survey on Ugc can be found in many places, for example in [22].

The Exponential Time Hypothesis (Eth) In an attempt to provide a rigorous evidence that the complexity of k-Sat increases with increasing k, Impagliazzo and Paturi in [12] formulated the so-called Exponential Time Hypothesis (Eth) in the following manner. Letting $s_k = \inf\{\, \delta$: there exists $O^*(2^{\delta n})$ algorithm for solving k-Sat $\}$, Eth states that $s_k > 0$ for all $k \geq 3$, *i.e.*, k-Sat does not admit a sub-exponential time (*i.e.*, of time $O^*(2^{o(n)})$) algorithm[2]. Eth has significant implications for worst-case time-complexity of exact solutions of search problems, *e.g.*, see [13, 24].

1.3. Our Results

Let $G = (V, E)$ be the given graph. It is easy to see following the approach in Khuller *et al.* [17] that the problem of computing the strong metric dimension sdim(G) can be reduced to an instance of the (unweighted) set-cover problem giving a $O(\log|V|)$-approximation. In this communication, we show further improved results as summarized by the following theorem.

Theorem 1.1.
(a) Str-Met-Dim *admits the following type of algorithms:*

- *polynomial-time* 2*-approximation,*

[2] For two functions $f(x)$ and $g(x)$ of x, $f = o(g)$ provided $\lim_{x\to\infty} f(x)/g(x) = 0$.

- $O^*(2^{0.287\,n})$*-time exact computation algorithm, and*
- $O(1.2738^k + n\,k)$*-time exact computation algorithm where* $sdim(G) \leq k$*.*

(b) *Assuming that the unique games conjecture (* Ugc *) is true,* Str-Met-Dim *does not admit a polynomial-time* $(2-\varepsilon)$*-approximation for any constant* $0 < \varepsilon \leq 1$ *even if the given graph is restricted in the sense that*

(i) $diam(G) \leq 2$*, or*

(ii) *G is bipartite and* $diam(G) \leq 4$*.*

(c) *Assuming* P≠NP, Str-Met-Dim *does not admit a polynomial-time* $(10\sqrt{5}-21-\varepsilon)$*-approximation*[3] *for any constant* $0 < \varepsilon \leq 10\sqrt{5}-22$ *even if the given graph is restricted in the sense that*

(i) $diam(G) \leq 2$*, or*

(ii) *G is bipartite and* $diam(G) \leq 4$*.*

(d) *Assuming the exponential time hypothesis (*Eth*) is true, the following results hold for a graph G of n nodes:*

(i) *there is no* $O^*(2^{o(n)})$*-time algorithm for exactly computing* $sdim(G)$*, and*

(ii) *if* $sdim(G) \leq k$ *then there is no* $O^*(n^{o(k)})$*-time algorithm for exactly computing* $sdim(G)$*.*

1.4. Brief Remark on the Proof of Theorem 1.1

Our proof uses Theorem 2.1 whose proof is implicit in [18]. However, it is not the case that Theorem 2.1 can be simply "plugged in" to get a proof of our inapproximability results. Just because a problem can be written as a node cover problem (as in Fact 2.1) does not necessarily mean that it has the same inapproximability property for node cover since, for example, non-trivial special cases of node cover do admit efficient polynomial time solution. To show inapproximability we need to reduce appropriate "hard" instances of the node cover problem to that of computing sdim(G) (*i.e.*, a reduction in the opposite direction) and moreover such a polynomial-time reduction must be gap-preserving in an appropriate way (see [1, Section 10.1.3] for descriptions of gap-preserving reductions). For readers unfamiliar with gap-preserving reduction proof techniques, see the excellent survey by Arora and Lund in [1].

2. Proof of Theorem 1.1

The minimum node cover (Mnc) problem for a graph is defined as follows:

Instance:	an undirected graph $G = (V, E)$.
Valid Solution:	a set of nodes $V' \subseteq V$ such that $V' \cap \{u, v\} \neq \emptyset$ for every edge $\{u, v\} \in E$.
Objective:	*minimize* $\lvert V' \rvert$.
Related notation:	$\text{Mnc}(G) = \min_{\forall \{u,v\} \in E:\ V' \cap \{u,v\} \neq \emptyset} \left\{ \lvert V' \rvert \right\}$.

Let $G = (V, E)$ denote the input graph of n nodes. Let $\widehat{G}$ and $\widetilde{G}$ be two graphs obtained from G in the following manner:

[3]Note that $10\sqrt{5} - 21 \approx 1.36068 < 2$.

- $\widehat{G} = (V, \widehat{E})$ where $\{u, v\} \in \widehat{E}$ if and only if $u \neq v$ **and** $u \overset{s}{\leftrightsquigarrow} v$ is a maximal shortest path in G.

- $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ where $\widetilde{V}$ and $\widetilde{E}$ are obtained as follows:

  - Let $u_1, u_2, \ldots, u_\kappa$ be the nodes in G such that, for every u_i ($1 \leq i \leq \kappa$), there is a node $v_i \neq u_i$ in G with the property that $N(u_i) = N(v_i)$.
  - Let $\overline{G} = (V, \overline{E})$ be the (edge) complement of G, *i.e.*, $\{u, v\} \in \overline{E} \equiv \{u, v\} \notin E$.
  - Then, $\widetilde{V} = V \cup \{x_1, x_2, \ldots, x_\kappa, y\}$ where $x_1, x_2, \ldots, x_\kappa, y \notin V$, and $\widetilde{E} = \overline{E} \cup \left(\bigcup_{j=1}^{\kappa} \left\{ \{x_j, u_j\} \right\} \right) \cup \left(\bigcup_{y' \in \widetilde{V} \setminus \{y\}} \left\{ \{y', y\} \right\} \right)$.

We recall the following result implicit in [18].

Theorem 2.1. [18]
(a) $\mathrm{sdim}(G) = \textsc{Mnc}(\widehat{G})$, *and* $V' \subseteq V$ *is a valid solution of* Str-Met-Dim *on* G *if and only if* V' *is a valid solution of* Mnc *on* $\widehat{G}$*.*

(b) $\mathrm{diam}(\widetilde{G}) = 2$ *and* $\mathrm{sdim}(\widetilde{G}) = \kappa + \textsc{Mnc}(G)$.

A proof of Theorem 2.1 is implicit in [18]. For reader's benefit, we provide a self-contained proof of Theorem 2.1 in Appendix A using elementary graph theory.

Proof of Theorem 1.1(a)

Since $\mathrm{sdim}(G) = \textsc{Mnc}(\widehat{G})$, and both G and $\widehat{G}$ have the same number of nodes, the claim follows by applying known algorithms for node cover on $\widehat{G}$. More precisely,

- the 2-approximation follows from a well-known 2-approximation algorithm for Mnc [23, Theorem 1.3],

- the $O^*(2^{0.287\,n})$-time exact solution algorithm follows from the $O^*(2^{0.287\,n})$-time exact algorithm for maximum independent set[4] problem in [7], and

- the $O(1.2738^k + n\,k)$-time exact computation algorithm follows from the $O(1.2738^k + n\,k)$-time exact algorithm for minimum node cover of $\widehat{G}$ provided $\textsc{Mnc}(\widehat{G}) \leq k$ [3].

Proof of Theorem 1.1(b)

Consider the standard Boolean satisfiability problem (Sat) [9] and let Φ be an input instance of Sat. Our starting point is the following inapproximability result proved by Khot and Regev [16]:

> *Assuming* Ugc *is true, there exists a polynomial time algorithm that transforms a given instance* Φ *of* Sat *to an input instance graph* $G = (V, E)$ *of* Mnc *with* n *nodes such that, for any constant* $0 < \varepsilon < \frac{1}{4}$*, the following holds:*

$$(\star) \quad \begin{array}{rl} \textbf{\textit{(YES case)}} & \textit{if } \Phi \textit{ is satisfiable then } \textsc{Mnc}(G) \leq \left(\frac{1}{2} + \varepsilon\right) n, \textit{ and} \\ \textbf{\textit{(NO case)}} & \textit{if } \Phi \textit{ is not satisfiable then } \textsc{Mnc}(G) \geq (1 - \varepsilon)\, n. \end{array}$$

[4]Nodes *not* in an independent set form a *valid* solution of the node cover problem [9].

Consider such an instance G of MNC as generated by the above transformation. Let $k = 1 + \lfloor \log_2 n \rfloor$ and let $b(j) = b_{k-1}(j)\, b_{k-2}(j) \ldots b_1(j)\, b_0(j)$ be the binary representation of an integer $j \in \{1, 2 \ldots, n\}$ using *exactly* k bits (*e.g.*, if $n = 5$ then $b(3) = \overset{b_2(3)}{0}\ \overset{b_1(3)}{1}\ \overset{b_0(3)}{1}$). Let $u_1, u_2, \ldots, u_n$ be an arbitrary ordering of the nodes in V. We first construct the following graph $G^+ = (V^+, E^+)$ from G:

- $V^+ = V \cup V_1^+$ where $V_1^+ = \{v_1, v_2, \ldots, v_{k-1}, y\}$ is a set of k new nodes, and
- $E^+ = E \;\cup\; \left(\bigcup_{j=1}^{n} \left\{ \{u_j, v_\ell\} \mid b_\ell(j) = 1 \right\} \right) \;\cup\; \left(\bigcup_{j=1}^{k-1} \left\{ \{y, v_j\} \right\} \right)$.

Thus $|V^+| = n + k$ and $|E^+| < |E| + \frac{nk}{2} + k$. Now, note that:

- if $V' \subseteq V$ is a solution of MNC on G, then $V' \cup V_1^+$ is a solution of MNC on G^+, implying MNC$(G^+) \leq$ MNC$(G) + k$, and conversely,
- if $V' \subseteq V^+$ is a solution of MNC on G^+, then $V' \setminus V_1^+$ is a solution of MNC on G, implying MNC$(G) \leq$ MNC(G^+).

Combining the above inequalities with that in $(\star)$, we have

$(\star\star)$ **(YES case)** if Φ is satisfiable then MNC$(G^+) < \left(\frac{1}{2} + \varepsilon\right) n + \log_2 n + 1$, and
(NO case) if Φ is not satisfiable then MNC$(G^+) \geq (1 - \varepsilon)\, n$.

We now build the graph $\widetilde{G^+} = (\widetilde{V^+}, \widetilde{E^+})$ from G using the construction in Theorem 2.1**(b)**.

Claim 2.1.1. *No two nodes in $\widetilde{G^+}$ have the same neighborhood.*

Proof. The following careful case analysis proves the claim:

- For any $i \neq j$, since $b(i) \neq b(j)$, there exists an index t such that $b_t(i) \neq b_t(j)$, say $b_t(i) = 0$ and $b_t(j) = 1$. Thus, $\mathrm{N}(u_i) \neq \mathrm{N}(u_j)$ since $v_t \in \mathrm{N}(u_j)$ but $v_t \notin \mathrm{N}(u_i)$.
- Since $b(i) \neq 0$ for any i and $b(1), b(2), \ldots, b(n)$ are distinct binary numbers each of exactly k bits, for any $t \neq t'$ there is an index i such that $b_t(i) \neq b_{t'}(i)$, say $b_t(i) = 0$ and $b_{t'}(i) = 1$. Thus, $\mathrm{N}(v_t) \neq \mathrm{N}(v_{t'})$ since $u_i \in \mathrm{N}(v_{t'})$ but $u_i \notin \mathrm{N}(v_t)$.
- For any i and j, $\mathrm{N}(u_i) \neq \mathrm{N}(v_j)$ since $y \in \mathrm{N}(v_j)$ but $y \notin \mathrm{N}(u_i)$.
- For any i, $b(i) \neq 0$ and thus there exists an index j such that $b_j(i) = 1$. This implies $u_j \in \mathrm{N}(v_i)$ but $u_j \notin \mathrm{N}(y)$ and therefore $\mathrm{N}(v_i) \neq \mathrm{N}(y)$.
- Since G is a connected graph, for every node u_i there exists a node u_j such that $\left\{u_i, u_j\right\} \in E^+$. Thus, $u_j \in \mathrm{N}(u_i)$ but $u_j \notin \mathrm{N}(y)$, implying $\mathrm{N}(u_i) \neq \mathrm{N}(y)$.

□

By the above claim, $\kappa = 0$ and $\mathrm{sdim}(\widetilde{G^+}) =$ MNC(G^+) by Theorem 2.1**(b)**. Thus, setting $\varepsilon' = \varepsilon + \frac{\log_2 n + 1}{n}$ and noting that ε' can be any arbitrarily small constant since ε is an arbitrarily small constant, it follows from $(\star\star)$ that

$$(\star\star\star)\quad \begin{array}{l}\textbf{(YES case)} \text{ if } \Phi \text{ is satisfiable then } \mathrm{sdim}(\widetilde{G^+}) = \textsc{Mnc}(G^+) < \left(\frac{1}{2}+\varepsilon'\right)n, \text{ and} \\ \textbf{(NO case)} \text{ if } \Phi \text{ is not satisfiable then } \mathrm{sdim}(\widetilde{G^+}) = \textsc{Mnc}(G^+) \geq (1-\varepsilon')\,n.\end{array}$$

This proves Theorem 1.1**(b)(i)** since $\mathrm{diam}(\widetilde{G^+}) = 2$ by Theorem 2.1**(b)**.

To prove Theorem 1.1**(b)(ii)**, we modify the graph $\widetilde{G^+}$ to a new graph $G' = (V', E')$ by splitting every edge into a sequence of two edges, *i.e.*, for every edge $\{u, v\}$ in $\widetilde{G^+}$ we add a new node $x_{u,v}$ in G' and replace the edge $\{u, v\}$ by the two edges $\{u, x_{u,v}\}$ and $\{v, x_{u,v}\}$. Clearly G' is bipartite since all its cycles are of even length and $\mathrm{diam}(G') \leq 2\,\mathrm{diam}(\widetilde{G^+}) = 4$.

Claim 2.1.2. *$sdim(\widetilde{G^+}) = \textsc{Mnc}(\widehat{\widetilde{G^+}}) = \textsc{Mnc}(\widehat{G'}) = sdim(G')$.*

Proof. No maximal shortest path in G' ends at a node $x_{u,v}$ for any distinct pair of nodes u and v. Indeed, if a maximal shortest path $\mathcal{P}$ from some node z ends at some $x_{u,v}$, it must use one of the two edges $\{u, x_{u,v}\}$ and $\{v, x_{u,v}\}$, say $\{u, x_{u,v}\}$. Then adding the edge $\{v, x_{u,v}\}$ to the path $\mathcal{P}$ provide a shortest path between v and z, and thus $\mathcal{P}$ was not maximal. Using this and the construction in Theorem 2.1**(a)**, we have $\widehat{\widetilde{G^+}} = \widehat{G'}$ and therefore $\mathrm{sdim}(\widetilde{G^+}) = \textsc{Mnc}(\widehat{\widetilde{G^+}}) = \textsc{Mnc}(\widehat{G'}) = \mathrm{sdim}(G')$. □

As a result, the inapproximability result for $\mathrm{sdim}(\widetilde{G^+})$ directly translates to that for $\mathrm{sdim}(G')$, and concludes the proof.

Proof of Theorem 1.1(c)

The same proof as in **(b)** works provided, instead of the result in [16], our starting point is the following result shown by Dinur and Safra [6][5]:

> *Assuming P≠NP, there exists a polynomial time algorithm that transforms a given instance Φ of* Sat *to an input instance graph $G = (V, E)$ of* Mnc *with n nodes such that, for any constant $0 < \varepsilon < 16 - 8\sqrt{5}$ and for some $0 < \alpha < 2n$, the following holds:*

$$(\star)\quad \begin{array}{ll}\textit{(YES case)} & \textit{if } \Phi \textit{ is satisfiable then } \textsc{Mnc}(G) \leq \left(\frac{\sqrt{5}-1}{2}+\varepsilon\right)\alpha, \textit{ and} \\ \textit{(NO case)} & \textit{if } \Phi \textit{ is not satisfiable then } \textsc{Mnc}(G) \geq \left(\frac{71-31\sqrt{5}}{2}-\varepsilon\right)\alpha.\end{array}$$

Proof of Theorem 1.1(d) We first show how to prove Theorem 1.1**(d)(i)**. Suppose, for the sake of contradiction, that there does exist a $O^*(2^{o(n)})$-time algorithm that exactly computes $\mathrm{sdim}(G)$. We start with an instance Φ of 3-Sat having n variables and m clauses. The "sparsification lemma" in [13] proves the following result:

> *for every constant $\varepsilon > 0$, there is a constant $c > 0$ such that there exists a $O(2^{\varepsilon n})$-time algorithm that produces from Φ a set of t instances $\Phi_1, \ldots, \Phi_t$ of* 3-Sat *on these n variables with the following properties:*
>
> - $t \leq 2^{\varepsilon n}$,

[5] Note that $\left(\frac{71-31\sqrt{5}}{2}\right) / \left(\frac{\sqrt{5}-1}{2}\right) = 10\sqrt{5} - 21$.

- *each Φ_j is an instance of* 3-SAT *with $n_j \leq n$ variables and $m_j \leq cn$ clauses, and*
- *Φ is satisfiable if and only if at least one of $\Phi_1, \ldots, \Phi_t$ is satisfiable.*

For each such above-produced 3-SAT instance Φ_j, we now use the "classical textbook" reduction from 3-SAT to the node cover problem (*e.g.*, see [9, page 54]) producing an instance $G = (V, E)$ of MNC of $|V| = 3n_j + 2m_j \leq (3 + 2c)\,n$ nodes and $|E| = n_j + m_j \leq (1 + c)\,n$ edges such that Φ_j is satisfiable if and only if $\text{MNC}(G) = n_j + 2m_j$. Moreover, it is also easy to check that this classical reduction does *not* produce two nodes in V that have the *same* neighborhood. Thus, setting $\kappa = 0$ in Theorem 2.1**(b)** we get $\text{sdim}(\widetilde{G}) = \text{MNC}(G)$ where $\widetilde{G}$ is a graph with $\widetilde{n} = |\widetilde{V}| = |V| + 1 \leq (3 + 2c)\,n + 1$ nodes. By assumption, we can compute $\text{sdim}(\widetilde{G})$ in $O^*(2^{o(\widetilde{n})})$ time, and and consequently $\text{MNC}(G)$ in $O^*(2^{o(n)})$ time, which leads us to decide in $O^*(2^{o(n)})$ time if Φ_j is satisfiable. Since $t \leq 2^{\varepsilon n}$ for any constant $\varepsilon > 0$, this provides a $O^*(2^{o(n)})$-time algorithm for 3-SAT, contradicting ETH.

To prove Theorem 1.1**(d)(ii)** suppose again, for the sake of contradiction, that there exists a $O^*(n^{o(k)})$-time algorithm for exactly computing $\text{sdim}(G)$ if $\text{sdim}(G) \leq k$. Our proof is very similar to the previous one, but this time we start with the following lower bound result on parameterized complexity (*e.g.*, see [5, Theorem 14.21]):

> *assuming* ETH *to be true, if* $\text{MNC}(G) \leq k$ *then there is* no $O^*(n^{o(k)})$*-time algorithm for exactly computing* $\text{MNC}(G)$.

Using the encoding as described in part **(b)** of this proof with the corresponding Claim 2.1.1, we can set $\kappa = 0$ in Theorem 2.1**(b)** to obtain the graph $\widetilde{G^+} = (\widetilde{V^+}, \widetilde{E^+})$ such that $\widetilde{n^+} = |\widetilde{V^+}| = |V| + (1 + \lfloor \log_2 n \rfloor) + 1 = n + \lfloor \log_2 n \rfloor + 2$ and $\text{sdim}(\widetilde{G^+}) = \text{MNC}(G)$. By our assumption, we can compute $\text{sdim}(\widetilde{G^+})$ in $O^*\Big((\widetilde{n^+})^{o(k)}\Big)$-time algorithm if $\text{sdim}(G) \leq k$. This then provides an algorithm running in $O^*\Big((\widetilde{n^+})^{o(k)}\Big) = O^*(n^{o(k)})$ time if $\text{MNC}(G) = \text{sdim}(G) \leq k$, contradicting ETH.

3. Conclusion

In this communication we have shown that the worst-case computational complexity for computing the strong metric dimension for many graphs behaves in a manner similar to the minimum node cover problem. However, several interesting computational complexity questions still remain open, such as the following.

- Does the $(2 - \varepsilon)$-inapproximability result for computing $\text{sdim}(G)$ hold even when G is bipartite and $\text{diam}(G) \leq 3$?
- Are there interesting non-trivial classes of graphs for which $\text{sdim}(G)$ can be computed in polynomial time ?
- In the context of kernelization for parameterized algorithms (*e.g.*, see [5]), is there a linear kernel for STR-MET-DIM?

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Appendix A. Proof of Theorem 2.1

A proof of Theorem 2.1 is implicit in [18]. For the benefit of the reader, we provide a self-contained proof of Theorem 2.1 here using elementary graph theory.

(a) Let $u \overset{s}{\leftrightsquigarrow} v$ be a maximal shortest path in G. Suppose that we select neither u nor v in a solution of solution of STR-MET-DIM on G. Then there exists no node x in our solution of STR-MET-DIM on G such that $x \blacktriangleright \{u, v\}$, implying our solution of STR-MET-DIM on G is *not* a valid solution and thereby showing $\mathrm{sdim}(G) \geq \textsc{Mnc}(\widehat{G})$. To prove $\mathrm{sdim}(G) \leq \textsc{Mnc}(\widehat{G})$, suppose that we select at least one end-point of every maximal shortest path in G. Consider any pair of nodes u and v. If at least one of u or v, say u, is selected in a solution of STR-MET-DIM on G, then

$u \blacktriangleright \{u, v\}$. Otherwise, $u \overset{s}{\leftrightsquigarrow} v$ is *not* a maximal shortest path, and let $x \overset{s}{\leftrightsquigarrow} y$ be a maximal shortest path containing u and v. Then, we have selected at least one of x or y, say x, in a solution of Str-Met-Dim on G, and $x \blacktriangleright \{u, v\}$.

(b) It follows from the construction of $\widetilde{G}$ that $\text{diam}(\widetilde{G}) = 2$ since any pair of nodes has a shortest path of length at most 2 between them via y. Note that, for any pair of nodes u and v, $\mathrm{N}(u) = \mathrm{N}(v)$ in G if and only if $\mathrm{N}(u) = \mathrm{N}(v)$ in $\overline{G}$. To show $\text{sdim}(\widetilde{G}) \leq \kappa + \text{Mnc}(G)$, let $S \subset V$ be the set of nodes in a minimum node cover of G of cardinality $\text{Mnc}(G)$. Consider the set of $\kappa + \text{Mnc}(G)$ nodes in $S' = S \cup \{x_1, x_2, \ldots, x_\kappa\}$ as a possible solution of Str-Met-Dim on $\widetilde{G}$. To show that this is indeed a valid solution, consider any pair of nodes u and v in $\widetilde{G}$. Then the following simple case analysis suffices:

- Suppose that at least one of u and v is x_i for some i. Then, $S' \ni x_i \blacktriangleright \{u, v\}$.
- Otherwise, suppose that one of u and v, say u, is y (and thus $v \in V$). Select a node $x_i \in S'$ such that $\{x_i, v\} \notin \widetilde{E}$. Then the shortest path of length 2 from x_i to v formed by the edges $\{x_i, y\}$ and $\{y, v\}$ shows that $S' \ni x_i \blacktriangleright \{u, v\}$.
- Otherwise, if $\{u, v\} \in E$ then at least one of u and v, say u, is in S' and $u \blacktriangleright \{u, v\}$.
- Otherwise, $\{u, v\} \notin E$. Thus, $\{u, v\} \in \widetilde{E}$. If at least one of u and v, say u, is in S then $u \in S'$ and $u \blacktriangleright \{u, v\}$. Otherwise, both of u and v are not in S, and there are the following two sub-cases to consider.
  - At least one of u and v, say u, is u_i for some i. Then the shortest path of length 2 from x_i to v formed by the edges $\{x_i, u_i\}$ and $\{u_i, v\}$ shows that $S' \ni x_i \blacktriangleright \{u, v\}$.
  - Otherwise, $\mathrm{N}(u) \neq \mathrm{N}(v)$ in G, which implies that there exists a node $u' \in V$ such that u' is adjacent to exactly one of u and v, say u. Thus, $\{u, u'\} \notin \widetilde{E}$ but $\{v, u'\} \in \widetilde{E}$. Note that $u \notin S$ and $\{u, u'\} \in E$ implies u' is in S. Then the shortest path of length 2 from u' to u formed by the edges $\{u', v\}$ and $\{v, u\}$ shows that $S' \ni u' \blacktriangleright \{u, v\}$.

To show $\text{sdim}(\widetilde{G}) \geq \kappa + \text{Mnc}(G)$, let $S' \subset \widetilde{V}$ be the set of $\text{sdim}(\widetilde{G})$ nodes in an optimal solution of Str-Met-Dim on $\widetilde{G}$. Consider the set of nodes in $S = S' \setminus \{x_1, x_2, \ldots, x_\kappa, y\}$ as a possible solution of the node cover problem of G. We first show that S is in fact a valid node cover of G. Since $\text{diam}(\widetilde{G}) = 2$, any shortest path in G is of length at most 2. Consider an edge $\{u, v\} \in E$ and suppose that both u and v are not in S (and thus also not in S'). Since $\{u, v\} \notin \widetilde{E}$, the length of any shortest path between u and v is exactly 2, and thus no node $x \in \widetilde{V} \setminus \{u, v\}$ can strongly resolve the pair of nodes u and v, resulting in a contradiction that S' is a solution of Str-Met-Dim on $\widetilde{G}$. Thus, S is a node cover of G and $\text{Mnc}(G) \leq |S|$. To show that $|S| = |S'| - \kappa$, note that:

- Every x_i must belong to S' since otherwise no node in S' can strongly resolve the pair of nodes x_i and x_j for any $j \neq i$.
- Since every x_i belongs to S', the node y does not need to belong to S'.