Node Expansions and Cuts in Gromov-hyperbolic Graphs*

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Joint work with

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- Nasim Mobasheri (UIC)
- Farzaneh Yahyanejad

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Outline of talk

Graph-theoretical analysis leads to useful insights for many complex systems, such as

- **World-Wide Web**
- **social network of jazz musicians**
- **metabolic networks**
- **protein-protein interaction networks**

Examples of useful network measures for such analyses

- **degree based** , *e.g.*
  - **maximum/minimum/average degree, degree distribution,** ......
- **connectivity based** , *e.g.*
  - **clustering coefficient, largest cliques or densest sub-graphs,** ......
- **geodesic based** , *e.g.*
  - **diameter, betweenness centrality,** ......
- **other more complex measures**

network measure for this talk

Gromov-hyperbolicity measure δ

- **originally proposed by Gromov in 1987 in the context of group theory**
  - **observed that many results concerning the fundamental group of a Riemann surface hold true in a more general context**
  - **defined for infinite continuous metric space via properties of geodesics**
  - **can be related to standard scalar curvature of Hyperbolic manifold**
- **adopted to finite graphs using a 4-node condition or equivalently using thin triangles**

Are there real-world networks that are hyperbolic?

Yes, for example:

- Preferential attachment networks were shown to be scaled hyperbolic
  - [Jonckheere and Lohsoonthorn, 2004; Jonckheere, Lohsoonthorn and Bonahon, 2007]
- Networks of high power transceivers in a wireless sensor network were empirically observed to have a tendency to be hyperbolic
  - [Ariaei, Lou, Jonckeere, Krishnamachari and Zuniga, 2008]
- Communication networks at the IP layer and at other levels were empirically observed to be hyperbolic
  - [Narayan and Saniee, 2011]
- Extreme congestion at a very limited number of nodes in a very large traffic network was shown to be caused due to hyperbolicity of the network together with minimum length routing
  - [Jonckheerea, Loua, Bonahona and Baryshnikova, 2011]
- Topology of Internet can be effectively mapped to a hyperbolic space
  - [Bogun, Papadopoulos and Krioukov, 2010]

Motivation

Effect of δ on expansion and cut-size

Standard practice to investigate/categorize computational complexities of combinatorial problems in terms of ranges of topological measures:

- **Bounded-degree graphs are known to admit improved approximation as opposed to their arbitrary-degree counter-parts for many graph-theoretic problems.**
- **Claw-free graphs are known to admit improved approximation as opposed to general graphs for graph-theoretic problems such as the maximum independent set problem.**

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- **Claw-free graphs are known to admit improved approximation as opposed to general graphs for graph-theoretic problems such as the maximum independent set problem.**

Motivation for this paper: Effect of δ on expansion and cut-size

- **What is the effect of δ on expansion and cut-size bounds on graphs ?**
- **For what asymptotic ranges of values of δ can these bounds be used to obtain improved approximation algorithms for related combinatorial problems ?**

Outline of talk

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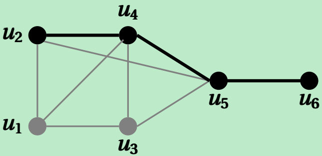
Graphs, geodesics and related notations

$G=(V,E)$	connected undirected graph of $n \geq 4$ nodes
$u \overset{\mathscr{P}}{\leftrightsquigarrow} v$	path $\mathscr{P} \equiv (\underbrace{u_0}_{=u}, u_1, \ldots, u_{k-1}, \underbrace{u_k}_{=v})$ between nodes u and v
$\ell(\mathscr{P})$	length (number of edges) of the path $u \overset{\mathscr{P}}{\leftrightsquigarrow} v$
$u_i \overset{\mathscr{P}}{\leftrightsquigarrow} u_j$	sub-path $(u_i, u_{i+1}, \ldots, u_j)$ of $\mathscr{P}$ between nodes u_i and u_j
$u \overset{\mathfrak{s}}{\leftrightsquigarrow} v$	a shortest path between nodes u and v
$d_{u,v}$	length of a shortest path between nodes u and v



$u_2 \overset{\mathscr{P}}{\leftrightsquigarrow} u_6$ is the path $\mathscr{P} \equiv (u_2, u_4, u_5, u_6)$

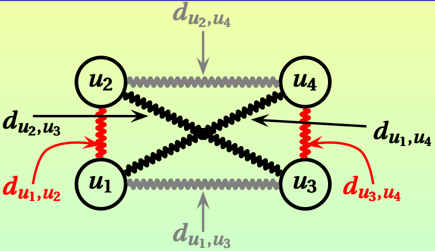
$\ell(\mathscr{P}) = 3$

$d_{u_2,u_6} = 2$

Basic definitions and notations

4 node condition

Consider four nodes u_1, u_2, u_3, u_4 and the six shortest paths among pairs of these nodes



Basic definitions and notations

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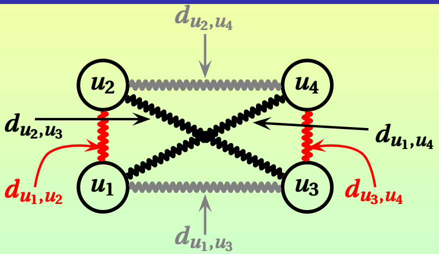
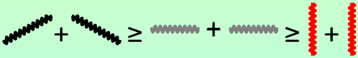


Assume, without loss of generality, that

$$\underbrace{d_{u_1,u_4} + d_{u_2,u_3}}_{=L} \geq \underbrace{d_{u_1,u_3} + d_{u_2,u_4}}_{=M} \geq \underbrace{d_{u_1,u_2} + d_{u_3,u_4}}_{=S}$$



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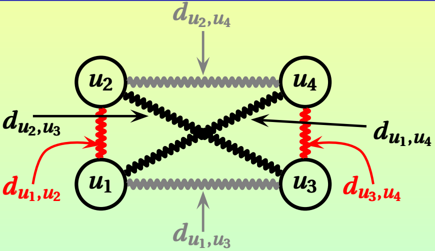

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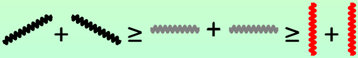
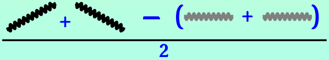


Let $\delta_{u_1,u_2,u_3,u_4} = \frac{L-M}{2}$



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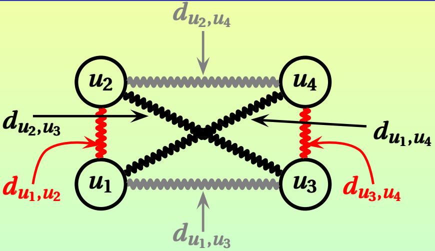

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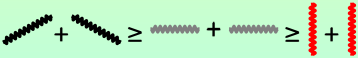

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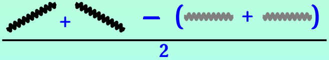

Definition (hyperbolicity of G)

$$\delta(G) = \max_{u_1,u_2,u_3,u_4} \left\{ \delta_{u_1,u_2,u_3,u_4} \right\}$$

Equivalent definition via geodesic triangles

(up to a constant multiplicative factor)

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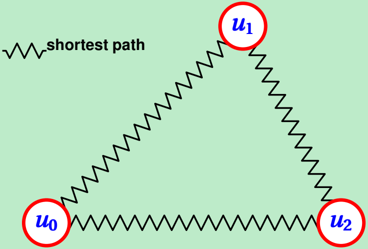

for every ordered triple of shortest paths

$$u_0 \overset{\mathfrak{s}}{\leftrightsquigarrow} u_1,\ u_0 \overset{\mathfrak{s}}{\leftrightsquigarrow} u_2,\ u_1 \overset{\mathfrak{s}}{\leftrightsquigarrow} u_2$$

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Equivalent definition via geodesic triangles

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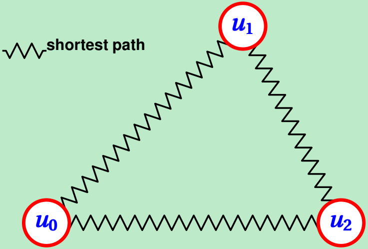

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$u_0 \overset{\mathfrak{s}}{\leftrightsquigarrow} u_1 \cup u_1 \overset{\mathfrak{s}}{\leftrightsquigarrow} u_2$

Basic definitions and notations

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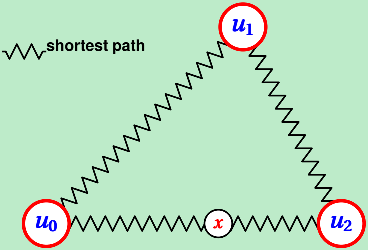

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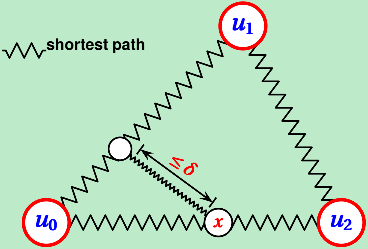

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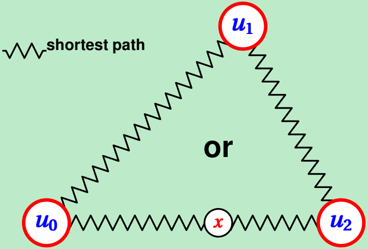

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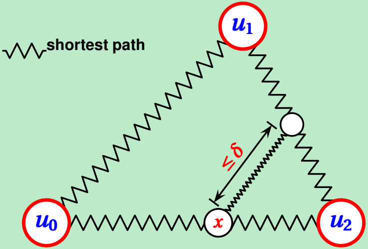

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Basic definitions and notations

Hyperbolic graphs

Definition (Δ-hyperbolic graphs)

G **is Δ-hyperbolic provided** $\delta(G) \leq \Delta$

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If Δ is a constant independent of graph parameters, then a Δ-hyperbolic graph is simply called a hyperbolic graph

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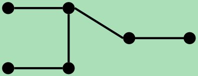
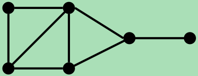
Example (Hyperbolic and non-hyperbolic graphs)

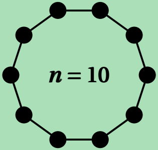
Tree: $\Delta(G) = 0$ hyperbolic graph



Chordal (triangulated) graph: $\Delta(G) = 1/2$ hyperbolic graph



Simple cycle: $\Delta(G) = \lceil n/4 \rceil$ non-hyperbolic graph



Computation of $\delta(G)$

- Trivially in $O(n^4)$ time
  - Compute all-pairs shortest paths — Floyd–Warshall algorithm $O(n^3)$ time
  - For each combination u_1, u_2, u_3, u_4, compute δ_{u_1,u_2,u_3,u_4} — $O(n^4)$ time
- Via $(\max, \min)$ matrix multiplication [Fournier, Ismail and Vigneron, 2015]
  - exactly in $O(n^{3.69})$ time
  - 2-approximation in in $O(n^{2.69})$ time

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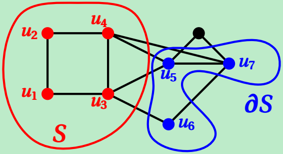
Outline of talk

Definition (Node expansion ratio $h(S)$ (n is number of nodes))



witness for $h(S)$

$S = \{u_1, u_2, u_3, u_4\}$

$$|S| \leq \frac{n}{2} \quad h(S) = \frac{|\partial(S)|}{|S|}$$

$\partial(S) = \{u_5, u_6, u_7\}$

$$\mathfrak{h} = \min_{|S| \leq \frac{n}{2}} \left\{ h(S) \right\}$$

Theorem (Nested Family of Witnesses for Node Expansion)

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Input:

- graph $G=(V,E)$ with n nodes and m edges — undirected unweighted
- maximum node degree d
- hyperbolicity δ
- two node p,q with $\Delta = d_{p,q}$ — distance between p and q

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For any constant $0<\mu<1$, there exists at least $t=\max\left\{\frac{\Delta^{\mu}}{56\log d},1\right\}$ subsets of nodes $\emptyset\subset S_1\subset S_2\subset\cdots\subset S_t\subset V$, each of at most $\frac{n}{2}$ nodes, with the following properties:

- $\forall\, j\in\{1,2,\dots,t\}:\ h\left(S_j\right)\leq\min\left\{\frac{8\ln\left(\frac{n}{2}\right)}{\Delta},\ \max\left\{\left(\frac{1}{\Delta}\right)^{1-\mu},\ \frac{500\ln n}{\Delta\, 2^{\frac{\Delta^{\mu}}{28\delta\log_2(2d)}}}\right\}\right\}$
- All the subsets can be found in a total of $O\left(n^3\log n+mn^2\right)$ time
- Either all the subsets contain node p, or all of them contain node q

Illustration of asymptotics of the expansion bound

$$\min\left\{ \frac{8\ln\left(\frac{n}{2}\right)}{\Delta}, \max\left\{ \left(\frac{1}{\Delta}\right)^{1-\mu}, \frac{500\ln n}{\Delta\, 2^{\frac{\Delta^{\mu}}{28\delta\log_2(2d)}}} \right\} \right\}$$

n **nodes, maximum degree** d

Effect of δ on Expansions in δ-hyperbolic Graphs

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n nodes, maximum degree $d \Rightarrow$ diameter $\geq \frac{\log_2 n}{\log_2 d}$

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$\Rightarrow \exists$ nodes p, q such that $\Delta = d_{p,q} = \frac{\log_2 n}{\log_2 d}$

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$$\max\left\{ \left(\frac{\log_2 d}{\log_2 n}\right)^{1-\mu}, \frac{500\log_2 d}{\frac{\log_2^{\mu} n}{2^{28\delta\log_2^{1+\mu}(2d)}}} \right\}$$

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First component of the bound

- $O\left(1/\log^{1-\mu} n\right)$ for fixed d
- $\Omega(1)$ only when $d = \Omega(n)$

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Illustration of asymptotics of the expansion bound

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Second component of the bound

suppose G is hyperbolic of constant maximum degree

***i.e.*, $\delta = O(1)$ and $d = O(1)$**

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$$\frac{500\log_2 d}{\frac{\log_2^{\mu} n}{2^{28\delta \log_2^{1+\mu}(2d)}}} = O\left(\frac{1}{2^{O(1)\log^{\mu} n}}\right) = O\left(\frac{1}{\text{polylog}(n)}\right)$$

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Second component of the bound

suppose G is hyperbolic but maximum degree d is varying
***i.e.*, $\delta = O(1)$ and d is variable**

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Effect of δ on Expansions in δ-hyperbolic Graphs

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$$\frac{500 \log_2 d}{ 2^{ \frac{\log_2^{\mu} n}{28\delta \log_2^{1+\mu}(2d)} } } = O\left(\frac{\log d}{2^{O(1) \log^{\mu} n / \log^{1+\mu} d}} \right) = O\left(\frac{\log d}{\text{polylog}(n)^{\frac{1}{\log^{1+\mu} d}}} \right)$$

$\underbrace{\Omega(1)}_{\text{expander}}$ **only if** $d > 2^{\Omega\left(\sqrt{\frac{\log\log n}{\log\log\log n}} \right)}$

Illustration of asymptotics of the expansion bound

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Second component of the bound

suppose G is not hyperbolic but had constant maximum degree
***i.e.*, $d = O(1)$ and δ is variable**

Effect of δ on Expansions in δ-hyperbolic Graphs

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$$\frac{500\log_2 d}{2^{\frac{\log_2^{\mu} n}{28\delta\log_2^{1+\mu}(2d)}}} = O\left(\frac{1}{2^{O(1)\frac{\log^{\mu} n}{\delta}}}\right)$$

$\underbrace{\Omega(1)}_{\text{expander}}$ **only if** $\delta = \Omega\left(\log^{\mu} n\right)$

Effect of δ on Expansions in δ-hyperbolic Graphs

Family of Witnesses for Node Expansion With Limited Mutual Overlaps

Theorem (**Witnesses for Node Expansion with Limited Overlaps)**

Effect of δ on Expansions in δ-hyperbolic Graphs

Family of Witnesses for Node Expansion With Limited Mutual Overlaps

Theorem (Witnesses for Node Expansion with Limited Overlaps)

Input:

- ▷ **graph $G=(V,E)$ with n nodes and m edges** — undirected unweighted
- ▷ **maximum node degree d**
- ▷ **hyperbolicity δ**
- ▷ **two node p,q with $\Delta = d_{p,q}$** — distance between p and q

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For any constant $0<\mu<1$ and any integer $\tau < \frac{\Delta}{\left(42\delta \log_2(2d)\log_2(2\Delta)\right)^{1/\mu}}$, there exists $\tau/4$ distinct collections of subsets of nodes $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{\tau/4} \subset 2^V$ such that:

Theorem (Witnesses for Node Expansion with Limited Overlaps)

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For any constant $0<\mu<1$ and any integer $\tau < \frac{\Delta}{\left(42\,\delta\,\log_2(2d)\,\log_2(2\Delta)\right)^{1/\mu}}$, there exists $\tau/4$ distinct collections of subsets of nodes $\mathscr{F}_1,\mathscr{F}_2,\dots,\mathscr{F}_{\tau/4}\subset 2^V$ such that:

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Effect of δ on Expansions in δ-hyperbolic Graphs

Family of Witnesses for Node Expansion With Limited Mutual Overlaps

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- All subsets in each $\mathcal{F}_j$ can be found in a total of $O\left(n^3\log n+mn^2\right)$ time

Illustration of the "limited overlap" bound

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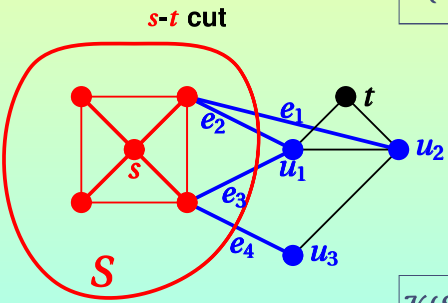
  - every pair of subsets from different families
    - is disjoint
    - or has at least $\Omega\left((\log_2 n)^{1/2}\right)$ private nodes

Effect of δ on Cuts in δ-hyperbolic Graphs

Definition of s-t cut and size of s-t cut



cut edges

$$\mathcal{E}(S,s,t) = \left\{ e_1, e_2, e_3, e_4 \right\}$$

$$\mathcal{V}(S,s,t) = \left\{ u_1, u_2, u_3 \right\}$$

cut nodes

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Input:

- graph $G=(V,E)$ with n nodes and m edges undirected unweighted
- d is maximum degree of any node except s, t and
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- hyperbolicity δ
- two node s,t with $d_{s,t} > 48\delta + 8\delta\log n$ distance between s and t is at least logarithmic in n

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  - each such cut has at most $d^{12\delta+1}$ cut edges

Outline of talk

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Network Design Application: Minimizing Bottleneck Edges

[Assadi *et al.*, 2014; Omran, Sack and Zarrabi-Zadeh, 2013; Zheng, Wang, Yang and Yang, 2010]

applications in several communication network design problems

Problem (**Unweighted Uncapacitated Minimum Vulnerability (UUMV)**)

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Goal

- find κ paths between s and t
- minimize number of shared edges

Minimizing Bottleneck Edges: Known results

- UUMV does not admit a $2^{\log^{1-\varepsilon} n}$-approximation for any constant $\varepsilon > 0$ unless NP⊆DTIME$\left(n^{\log\log n}\right)$ even if $r = 1$
- UUMV admits a $\left\lfloor \frac{\kappa}{r+1} \right\rfloor$-approximation
  - However, no non-trivial approximation of UUMV that depends on m and/or n only is currently known
- For $r = 1$, UUMV admits a $\min\left\{n^{\frac{3}{4}}, m^{\frac{1}{2}}\right\}$-approximation

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Minimizing Bottleneck Edges: Our result

Lemma (**Approximation of UUMV for δ-hyperbolic graphs)**

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Remark

- Lemma provides improved approximation as long as $\delta = o\left(\frac{\log n}{\log d}\right)$
- Our approximation ratio is independent of the value of κ
- $\delta = \Omega\left(\frac{\log n}{\log d}\right)$ allows expander graphs for which UUMV is expected to be harder to approximate

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Proof strategy overview

- **Define a new more general problem:**
  edge hitting set problem for size constrained cuts (EHSSC)
- **Show that UUMV has "similar" approximability properties as EHSSC**
- **Provide approximation algorithm for EHSSC using "family of cuts" lemma**

Small Set Expansion Problem

[Gandhi and Kortsarz, 2015; Bansal *et al.*, 2011; Raghavendra and Steurer, 2010; Arora, Barak and Steurer, 2010; ....]

application: studying Unique Games Conjecture

Problem (Small Set Expansion (SSE))

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For a subset of nodes S: $\Phi(S) = \dfrac{\text{number of cut edges from } S \text{ to } V\setminus S}{\text{sum of degrees of the nodes in } S}$

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- G has subset S of $\leq \zeta n$ nodes, for some constant $0 < \zeta < \frac{1}{2}$, such that $\Phi(S) \leq \varepsilon$ for some constant $0 < \varepsilon \leq 1$

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Goal

Find a subset S' of $\leq \zeta n$ nodes such that

- $\Phi(S') \leq \eta \varepsilon$ for some "universal constant" $\eta > 0$

Summary of “what is known” about SSE

- computing a good approximation of SSE seems to be quite hard
  - approximation ratio of algorithm in [Raghavendra, Steurer and Tetali, 2010] deteriorates proportional to $\sqrt{\log\left(\frac{1}{\zeta}\right)}$
  - $O(1)$-approximation in [Bansal *et al.*, 2011] works only if the graph excludes two specific minors
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Algorithmic Applications

Small Set Expansion Problem

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Our result

polynomial time solution of SSE for δ-hyperbolic graphs
when δ is sub-logarithmic and d is sub-linear

Lemma

SSE can be solved in polynomial time provided d and δ satisfy:

$$d \leq 2^{\log^{\frac{1}{3}-\rho} n} \text{ and } \delta \leq \log^{\rho} n \text{ for some constant } 0 < \rho < \tfrac{1}{3}$$

Outline of talk

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Conclusion and Future Research

- **We provided the first known non-trivial bounds on expansions and cut-sizes for graphs as a function of hyperbolicity measure δ**
- **We showed how these bounds and their related proof techniques lead to improved algorithms for two related combinatorial problems**
- **We hope that these results sill stimulate further research in characterizing the computational complexities of related combinatorial problems over asymptotic ranges of δ**

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Some future research problems

- **Improve the bounds in our paper**
- **Can we get a polynomial-time solution of Unique Games Conjectire for some asymptotic ranges of δ ?**
  - **Obvious recursive approach encounters a hurdle since hyperbolicity is not a hereditary property, *i.e.*, removal of nodes or edges may change δ sharply**
- **Can our bounds on expansions and cut-sizes be used to get an improved approximation for the minimum multicut problem for $\delta = o(\log n)$?**

Thank you for your attention

Questions??