On optimal approximability results for computing the strong metric dimension

Bhaskar DasGupta

Department of Computer Science
University of Illinois at Chicago
Chicago, IL 60607
bdasgup@uic.edu

June 24, 2015

Joint work with

Nasim Mobasheri (UIC)

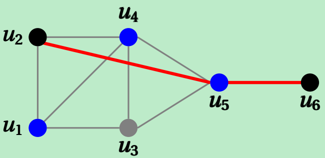
UIC

Outline of talk

1. Introduction
2. Main result of this talk
3. Brief discussion of proof techniques

Basic notations

- $\text{Nbr}(u)$: set of neighbors of node u
- $u \overset{s}{\leftrightsquigarrow} v$: a shortest path between nodes u and v
- $d_{u,v}$: length (number of edges) of $u \overset{s}{\leftrightsquigarrow} v$
- $\text{diam}(G) = \max_{u,v} \left\{ d_{u,v} \right\}$: diameter of graph G



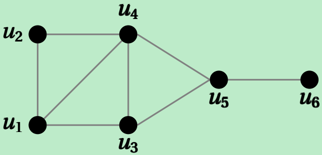
$\text{Nbr}(u_2) = \{u_1, u_4, u_5\}$

$u_2 \overset{s}{\leftrightsquigarrow} u_6$ is the path $u_2 - u_5 - u_6$

$d_{u_2,u_6} = 2$

Basic notations

- Nbr(u) : set of neighbors of node u
- $u \overset{s}{\leftrightsquigarrow} v$: a shortest path between nodes u and v
- $d_{u,v}$: length (number of edges) of $u \overset{s}{\leftrightsquigarrow} v$
- $\text{diam}(G) = \max_{u,v} \left\{ d_{u,v} \right\}$: diameter of graph G

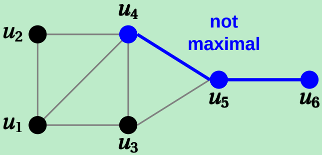


Definition (maximal shortest path)

$u \overset{s}{\leftrightsquigarrow} v$ is maximal if it is not properly included inside another shortest path

Basic notations

- $\mathrm{Nbr}(u)$: set of neighbors of node u
- $u \overset{s}{\leftrightsquigarrow} v$: a shortest path between nodes u and v
- $d_{u,v}$: length (number of edges) of $u \overset{s}{\leftrightsquigarrow} v$
- $\mathrm{diam}(G) = \max_{u,v} \left\{ d_{u,v} \right\}$: diameter of graph G

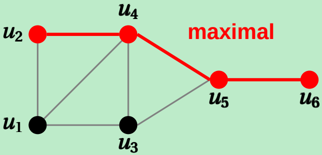


Definition (maximal shortest path)

$u \overset{s}{\leftrightsquigarrow} v$ is maximal if it is not properly included inside another shortest path

Basic notations

- $\text{Nbr}(u)$: set of neighbors of node u
- $u \overset{s}{\leftrightsquigarrow} v$: a shortest path between nodes u and v
- $d_{u,v}$: length (number of edges) of $u \overset{s}{\leftrightsquigarrow} v$
- $\text{diam}(G) = \max_{u,v} \left\{ d_{u,v} \right\}$: diameter of graph G



Definition (maximal shortest path)

$u \overset{s}{\leftrightsquigarrow} v$ is maximal if it is not properly included inside another shortest path

Definition (node x strongly resolves pair of nodes u and v)

$x \triangleright \{u, v\}$ if and only if

v is on a shortest path between x and u $\quad x \overset{s}{\leftrightsquigarrow} v \overset{s}{\leftrightsquigarrow} u \quad$ $x = v$ is allowed

or

u is on a shortest path between x and v $\quad x \overset{s}{\leftrightsquigarrow} u \overset{s}{\leftrightsquigarrow} v \quad$ $x = u$ is allowed

Definition (node x strongly resolves pair of nodes u and v)

$x \triangleright \{u,v\}$ if and only if

v is on a shortest path between x and u $\quad x \overset{s}{\leftrightsquigarrow} v \overset{s}{\leftrightsquigarrow} u \quad$ $x = v$ is allowed

or

u is on a shortest path between x and v $\quad x \overset{s}{\leftrightsquigarrow} u \overset{s}{\leftrightsquigarrow} v \quad$ $x = u$ is allowed

Definition (strongly resolving set of nodes V' for G)

$V' \triangleright G$ if and only if

some node in V' strongly resolves every distinct pair of nodes of G

$$\forall\, u,v \in V\ \exists\, x \in V' :\ x \triangleright \{u,v\}$$

Problem of computing strong metric dimension

Problem name	STR-MET-DIM
Instance	undirected graph $G = (V, E)$
Valid Solution	set of nodes V' such that $V' \triangleright G$
Objective	minimize $\lvert V' \rvert$

Related notation

$$\text{sdim}(G) = \min_{V' \triangleright G} \left\{ \lvert V' \rvert \right\}$$

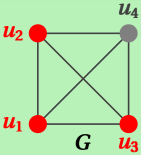
Problem of computing strong metric dimension

Problem name	STR-MET-DIM
Instance	undirected graph $G=(V,E)$
Valid Solution	set of nodes V' such that $V' \triangleright G$
Objective	minimize $\lvert V' \rvert$

Related notation

$$\text{sdim}(G) = \min_{V' \triangleright G} \left\{ \lvert V' \rvert \right\}$$

Example (**Illustration of** STR-MET-DIM)



$V' = \{u_1, u_2, u_3\}$

$\text{sdim}(G) = 3$

Basic concepts related to approximation algorithms

***e.g.*, see V. Vazirani, *Approximation Algorithms*, Springer-Verlag, 2001**

Minimization problem

Definition (ρ-approximation algorithm (algorithm with approximation ratio ρ))

- **runs in time polynomial in *size* of input**
- **produces solution with value $\leq \rho$ OPT** (OPT: optimum value)

Outline of talk

UIC

Main result of this talk

Theorem (Optimal approximability results for STR-MET-DIM)

- STR-MET-DIM **admits a polynomial-time 2-approximation algorithm**
- **Assuming that the unique games conjecture**[a] **is true,**

  STR-MET-DIM **does not admit a polynomial-time $(2-\varepsilon)$-approximation for any constant $\varepsilon > 0$**

  even if the given graph G satisfies

  - **$\text{diam}(G) \leq 2$, or**
  - **G is bipartite and $\text{diam}(G) \leq 4$**

[a] for definition of unique games conjecture, see S. Khot, *On the power of unique 2-Prover 1-Round games*, 34^{th} ACM Symposium on Theory of Computing, 2002

Outline of talk

1. Introduction
2. Main result of this talk
3. **Brief discussion of proof techniques**

Brief discussion of proof techniques

Minimum Node Cover problem (MNC)

Problem name MNC

Instance undirected graph $G = (V, E)$

Valid Solution set of nodes V' such that $V' \cap \{u, v\} \neq \emptyset$ for every edge $\{u, v\} \in E$

Objective minimize $|V'|$

Related notation

$$\text{MNC}(G) = \min_{\forall \{u,v\} \in E : V' \cap \{u,v\} \neq \emptyset} \left\{ |V'| \right\}$$

Brief discussion of proof techniques

Minimum Node Cover problem (MNC)

Problem name MNC

Instance undirected graph $G = (V, E)$

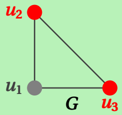
Valid Solution set of nodes V' such that $V' \cap \{u, v\} \neq \emptyset$ for every edge $\{u, v\} \in E$

Objective minimize $|V'|$

Related notation

$$\text{MNC}(G) = \min_{\forall \{u,v\} \in E : V' \cap \{u,v\} \neq \emptyset} \left\{ |V'| \right\}$$

Example (Illustration of MNC problem)



$V' = \{u_2, u_3\}$

$\text{MNC}(G) = 2$

Boolean satisfiability problem (SAT)

Problem name SAT

Instance

- n Boolean variables $x_1, x_2, \ldots, x_n$
- m clauses $C_1, C_2, \ldots, C_m$ over these variables

each clause is OR of some literals

literal is variable or negation of variable

Decision question is $\Phi \stackrel{\text{def}}{=} C_1 \wedge C_2 \wedge \cdots \wedge C_m$ satisfiable ?

can we set the variables such that Φ is true ?

Boolean satisfiability problem (SAT)

Problem name SAT

Instance
- n Boolean variables $x_1, x_2, \ldots, x_n$
- m clauses $C_1, C_2, \ldots, C_m$ over these variables

each clause is OR of some literals

literal is variable or negation of variable

Decision question is $\Phi \stackrel{\text{def}}{=} C_1 \wedge C_2 \wedge \cdots \wedge C_m$ satisfiable ?

can we set the variables such that Φ is true ?

Example (**Illustration of SAT**)

variables x_1, x_2, x_3, x_4

$$\Phi = \underbrace{(\neg x_1 \vee x_2)}_{C_1} \wedge \underbrace{(x_1 \vee \neg x_3 \vee x_4)}_{C_2} \wedge \underbrace{(x_3)}_{C_3}$$

Φ **is satisfiable**

$x_1 = x_2 = x_3 = x_4 = \text{TRUE}$

UIC

Brief discussion of proof techniques

Fact (S. Khot and O. Regev, *Vertex cover might be hard to approximate to within 2-ε*, Journal of Computer and System Sciences, 74(3), 335-349, 2008)

$\delta > 0$ any arbitrarily small constant
assume unique games conjecture is true

Instance Φ of SAT		Instance (graph) G of MNC with n nodes
Φ is satisfiable	→	$\text{MNC}(G) \leq \left(\frac{1}{2}+\delta\right)n$
Φ is NOT satisfiable	→ polynomial time transformation	$\text{MNC}(G) \geq (1-\delta)\,n$

Brief discussion of proof techniques for the result

Assuming that the unique games conjecture is true

STR-MET-DIM does not admit a polynomial-time $(2-\varepsilon)$-approximation even if $\mathrm{diam}(G) \leq 2$, or even if G is bipartite and $\mathrm{diam}(G) \leq 4$

$\delta > 0$ any arbitrarily small constant

assume unique games conjecture is true

Instance Φ of SAT	Graph G of MNC with n nodes	Graph $\tilde{G}$ of STR-MET-DIM with $n + \lfloor \log_2 n \rfloor + 1$ nodes $\mathrm{diam}(\tilde{G}) = 2$
Φ is satisfiable →	$\mathrm{MNC}(G) \leq \left(\frac{1}{2} + \delta\right) n$ →	$\mathrm{sdim}(\tilde{G}) < \left(\frac{1}{2} + \delta\right) n + \log_2 n + 1$
Φ is NOT satisfiable →	$\mathrm{MNC}(G) \geq (1-\delta)\, n$ →	$\mathrm{sdim}(\tilde{G}) \geq (1-\delta)\, n$
polynomial time transformation	polynomial time transformation	

Brief discussion of proof techniques for the result

Assuming that the unique games conjecture is true

STR-MET-DIM does not admit a polynomial-time $(2-\varepsilon)$-approximation even if $\mathrm{diam}(G) \leq 2$, or even if G is bipartite and $\mathrm{diam}(G) \leq 4$

$\delta > 0$ any arbitrarily small constant

assume unique games conjecture is true

$$\varepsilon = \frac{1}{2} + \delta + \frac{\log_2 n + 1}{n}$$

Instance Φ of SAT		Graph G of MNC with n nodes		Graph $\tilde{G}$ of STR-MET-DIM with $n + \lfloor \log_2 n \rfloor + 1$ nodes, $\mathrm{diam}(\tilde{G}) = 2$
Φ is satisfiable	→	$\mathrm{MNC}(G) \leq \left(\frac{1}{2} + \delta\right) n$	→	$\mathrm{sdim}\,(\tilde{G}) < \left(\frac{1}{2} + \delta\right) n + \log_2 n + 1$ $= \left(\frac{1}{2} + \varepsilon\right) n$
Φ is NOT satisfiable	→	$\mathrm{MNC}(G) \geq (1 - \delta)\, n$	→	$\mathrm{sdim}(\tilde{G}) \geq (1 - \delta)\, n$
	polynomial time transformation		polynomial time transformation	

Brief discussion of proof techniques for the result

NOT assuming unique games conjecture is true **but assuming P≠NP**

STR-MET-DIM **does not admit** a polynomial-time $(\overbrace{10\sqrt{5}-21}^{\approx 1.3606}-\varepsilon)$-approximation even if $\text{diam}(G)\leq 2$, or even if G is bipartite and $\text{diam}(G)\leq 4$

Brief discussion of proof techniques for the result

NOT assuming unique games conjecture is true **but assuming P≠NP**

STR-MET-DIM **does not admit** a polynomial-time $(\underbrace{10\sqrt{5}-21}_{\approx 1.3606}-\varepsilon)$-approximation even if $\mathrm{diam}(G)\leq 2$, or even if G is bipartite and $\mathrm{diam}(G)\leq 4$

$\delta>0$ any arbitrarily small constant

[Dinur and Safra, 2005]

Instance Φ of SAT

Graph G of MNC with n nodes

Graph $\tilde{G}$ of STR-MET-DIM with $n+\lfloor \log_2 n\rfloor+1$ nodes $\mathrm{diam}(\tilde{G})=2$

Φ is satisfiable → MNC$(G)\leq(10\sqrt{5}-21+\delta)\,n$ → $\mathrm{sdim}(\tilde{G})<(10\sqrt{5}-21+\delta)\,n+\log_2 n+1$

Φ is NOT satisfiable → MNC$(G)\geq(1-\delta)\,n$ → $\mathrm{sdim}(\tilde{G})\geq(1-\delta)\,n$

polynomial time transformation

polynomial time transformation

Final slide

Thank you for your attention

Questions??